

Project Gutenberg's Vector Analysis and Quaternions, by Alexander Macfarlane

This eBook is for the use of anyone anywhere at no cost and with almost no restrictions whatsoever. You may copy it, give it away or re-use it under the terms of the Project Gutenberg License included with this eBook or online at www.gutenberg.net

Title: Vector Analysis and Quaternions

Author: Alexander Macfarlane

Release Date: October 5, 2004 [EBook #13609]

Language: English

Character set encoding: TeX

*** START OF THIS PROJECT GUTENBERG EBOOK VECTOR ANALYSIS AND QUATERNIONS ***

Produced by David Starner, Joshua Hutchinson, John Hagerson, and the Project Gutenberg On-line Distributed Proofreaders.

MATHEMATICAL MONOGRAPHS.

EDITED BY
MANSFIELD MERRIMAN AND ROBERT S. WOODWARD.

No. 8.

VECTOR ANALYSIS

AND

QUATERNIONS.

BY

ALEXANDER MACFARLANE,
SECRETARY OF INTERNATIONAL ASSOCIATION FOR PROMOTING THE STUDY OF QUATERNIONS.

NEW YORK:

JOHN WILEY & SONS.

LONDON: CHAPMAN & HALL, LIMITED.

1906.

TRANSCRIBER'S NOTES: *This material was originally published in a book by Merriman and Woodward titled Higher Mathematics. I believe that some of the page number cross-references have been retained from that presentation of this material.*

I did my best to recreate the index.

MATHEMATICAL MONOGRAPHS.

EDITED BY

Mansfield Merriman and Robert S. Woodward.

Octavo. Cloth. \$1.00 each.

No. 1. History of Modern Mathematics.

By DAVID EUGENE SMITH.

No. 2. Synthetic Projective Geometry.

By GEORGE BRUCE HALSTED.

No. 3. Determinants.

By LAENAS GIFFORD WELD.

No. 4. Hyperbolic Functions.

By JAMES MCMAHON.

No. 5. Harmonic Functions.

By WILLIAM E. BYERLY.

No. 6. Grassmann's Space Analysis.

By EDWARD W. HYDE.

No. 7. Probability and Theory of Errors.

By ROBERT S. WOODWARD.

No. 8. Vector Analysis and Quaternions.

By ALEXANDER MACFARLANE.

No. 9. Differential Equations.

By WILLIAM WOOLSEY JOHNSON.

No. 10. The Solution of Equations.

By MANSFIELD MERRIMAN.

No. 11. Functions of a Complex Variable.

By THOMAS S. FISKE.

PUBLISHED BY

**JOHN WILEY & SONS, Inc., NEW YORK.
CHAPMAN & HALL, Limited, LONDON.**

Editors' Preface

The volume called Higher Mathematics, the first edition of which was published in 1896, contained eleven chapters by eleven authors, each chapter being independent of the others, but all supposing the reader to have at least a mathematical training equivalent to that given in classical and engineering colleges. The publication of that volume is now discontinued and the chapters are issued in separate form. In these reissues it will generally be found that the monographs are enlarged by additional articles or appendices which either amplify the former presentation or record recent advances. This plan of publication has been arranged in order to meet the demand of teachers and the convenience of classes, but it is also thought that it may prove advantageous to readers in special lines of mathematical literature.

It is the intention of the publishers and editors to add other monographs to the series from time to time, if the call for the same seems to warrant it. Among the topics which are under consideration are those of elliptic functions, the theory of numbers, the group theory, the calculus of variations, and non-Euclidean geometry; possibly also monographs on branches of astronomy, mechanics, and mathematical physics may be included. It is the hope of the editors that this form of publication may tend to promote mathematical study and research over a wider field than that which the former volume has occupied.

December, 1905.

Author's Preface

Since this Introduction to Vector Analysis and Quaternions was first published in 1896, the study of the subject has become much more general; and whereas some reviewers then regarded the analysis as a luxury, it is now recognized as a necessity for the exact student of physics or engineering. In America, Professor Hathaway has published a Primer of Quaternions (New York, 1896), and Dr. Wilson has amplified and extended Professor Gibbs' lectures on vector analysis into a text-book for the use of students of mathematics and physics (New York, 1901). In Great Britain, Professor Henrici and Mr. Turner have published a manual for students entitled Vectors and Rotors (London, 1903); Dr. Knott has prepared a new edition of Kelland and Tait's Introduction to Quaternions (London, 1904); and Professor Joly has realized Hamilton's idea of a Manual of Quaternions (London, 1905). In Germany Dr. Bucherer has published Elemente der Vektoranalysis (Leipzig, 1903) which has now reached a second edition.

Also the writings of the great masters have been rendered more accessible. A new edition of Hamilton's classic, the Elements of Quaternions, has been prepared by Professor Joly (London, 1899, 1901); Tait's Scientific Papers have been reprinted in collected form (Cambridge, 1898, 1900); and a complete edition of Grassmann's mathematical and physical works has been edited by Friedrich Engel with the assistance of several of the eminent mathematicians of Germany (Leipzig, 1894-). In the same interval many papers, pamphlets, and discussions have appeared. For those who desire information on the literature of the subject a Bibliography has been published by the Association for the promotion of the study of Quaternions and Allied Mathematics (Dublin, 1904).

There is still much variety in the matter of notation, and the relation of Vector Analysis to Quaternions is still the subject of discussion (see Journal of the Deutsche Mathematiker-Vereinigung for 1904 and 1905).

CHATHAM, ONTARIO, CANADA, December, 1905.

Contents

Editors' Preface	iii
Author's Preface	iv
1 Introduction.	1
2 Addition of Coplanar Vectors.	3
3 Products of Coplanar Vectors.	9
4 Coaxial Quaternions.	16
5 Addition of Vectors in Space.	21
6 Product of Two Vectors.	23
7 Product of Three Vectors.	28
8 Composition of Quantities.	32
9 Spherical Trigonometry.	37
10 Composition of Rotations.	44
Index	47
11 PROJECT GUTENBERG "SMALL PRINT"	

Article 1

Introduction.

By “Vector Analysis” is meant a space analysis in which the vector is the fundamental idea; by “Quaternions” is meant a space-analysis in which the quaternion is the fundamental idea. They are in truth complementary parts of one whole; and in this chapter they will be treated as such, and developed so as to harmonize with one another and with the Cartesian Analysis¹. The subject to be treated is the analysis of quantities in space, whether they are vector in nature, or quaternion in nature, or of a still different nature, or are of such a kind that they can be adequately represented by space quantities.

Every proposition about quantities in space ought to remain true when restricted to a plane; just as propositions about quantities in a plane remain true when restricted to a straight line. Hence in the following articles the ascent to the algebra of space is made through the intermediate algebra of the plane. Arts. 2–4 treat of the more restricted analysis, while Arts. 5–10 treat of the general analysis.

This space analysis is a universal Cartesian analysis, in the same manner as algebra is a universal arithmetic. By providing an explicit notation for directed quantities, it enables their general properties to be investigated independently of any particular system of coordinates, whether rectangular, cylindrical, or polar. It also has this advantage that it can express the directed quantity by a linear function of the coordinates, instead of in a roundabout way by means of a quadratic function.

The different views of this extension of analysis which have been held by independent writers are briefly indicated by the titles of their works:

- Argand, *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques*, 1806.
- Warren, *Treatise on the geometrical representation of the square roots of negative quantities*, 1828.
- Moebius, *Der barycentrische Calcul*, 1827.
- Bellavitis, *Calcolo delle Equipollenze*, 1835.

¹For a discussion of the relation of Vector Analysis to Quaternions, see *Nature*, 1891–1893.

- Grassmann, Die lineale Ausdehnungslehre, 1844.
- De Morgan, Trigonometry and Double Algebra, 1849.
- O'Brien, Symbolic Forms derived from the conception of the translation of a directed magnitude. Philosophical Transactions, 1851.
- Hamilton, Lectures on Quaternions, 1853, and Elements of Quaternions, 1866.
- Tait, Elementary Treatise on Quaternions, 1867.
- Hankel, Vorlesungen über die complexen Zahlen und ihre Functionen, 1867.
- Schlegel, System der Raumlehre, 1872.
- Hoüel, Théorie des quantités complexes, 1874.
- Gibbs, Elements of Vector Analysis, 1881–4.
- Peano, Calcolo geometrico, 1888.
- Hyde, The Directional Calculus, 1890.
- Heaviside, Vector Analysis, in “Reprint of Electrical Papers,” 1885–92.
- Macfarlane, Principles of the Algebra of Physics, 1891. Papers on Space Analysis, 1891–3.

An excellent synopsis is given by Hagen in the second volume of his “Synopsis der höheren Mathematik.”

Article 2

Addition of Coplanar Vectors.

By a “vector” is meant a quantity which has magnitude and direction. It is graphically represented by a line whose length represents the magnitude on some convenient scale, and whose direction coincides with or represents the direction of the vector. Though a vector is represented by a line, its physical dimensions may be different from that of a line. Examples are a linear velocity which is of one dimension in length, a directed area which is of two dimensions in length, an axis which is of no dimensions in length.

A vector will be denoted by a capital italic letter, as B ,¹ its magnitude by a small italic letter, as b , and its direction by a small Greek letter, as β . For example, $B = b\beta$, $R = r\rho$. Sometimes it is necessary to introduce a dot or a mark \angle to separate the specification of the direction from the expression for the magnitude;² but in such simple expressions as the above, the difference is sufficiently indicated by the difference of type. A system of three mutually rectangular axes will be indicated, as usual, by the letters i, j, k .

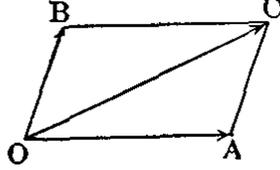
The analysis of a vector here supposed is that into magnitude and direction. According to Hamilton and Tait and other writers on Quaternions, the vector is analyzed into tensor and unit-vector, which means that the tensor is a mere ratio destitute of dimensions, while the unit-vector is the physical magnitude. But it will be found that the analysis into magnitude and direction is much more in accord with physical ideas, and explains readily many things which are difficult to explain by the other analysis.

A vector quantity may be such that its components have a common point of application and are applied simultaneously; or it may be such that its components are applied in succession, each component starting from the end of its

¹This notation is found convenient by electrical writers in order to harmonize with the Hospitalier system of symbols and abbreviations.

²The dot was used for this purpose in the author's Note on Plane Algebra, 1883; Kennelly has since used \angle for the same purpose in his electrical papers.

predecessor. An example of the former is found in two forces applied simultaneously at the same point, and an example of the latter in two rectilinear displacements made in succession to one another.



Composition of Components having a common Point of Application.—Let OA and OB represent two vectors of the same kind simultaneously applied at the point O . Draw BC parallel to OA , and AC parallel to OB , and join OC . The diagonal OC represents in magnitude and direction and point of application the resultant of OA and OB . This principle was discovered with reference to force, but it applies to any vector quantity coming under the above conditions.

Take the direction of OA for the initial direction; the direction of any other vector will be sufficiently denoted by the angle round which the initial direction has to be turned in order to coincide with it. Thus OA may be denoted by $f_1/\underline{0}$, OB by $f_2/\underline{\theta_2}$, OC by $f/\underline{\theta}$. From the geometry of the figure it follows that

$$f^2 = f_1^2 + f_2^2 + 2f_1f_2 \cos \theta_2$$

and

$$\tan \theta = \frac{f_2 \sin \theta_2}{f_1 + f_2 \cos \theta_2};$$

hence

$$OC = \sqrt{f_1^2 + f_2^2 + 2f_1f_2 \cos \theta_2} / \underline{\tan^{-1} \frac{f_2 \sin \theta_2}{f_1 + f_2 \cos \theta_2}}.$$

Example.—Let the forces applied at a point be $2/\underline{0^\circ}$ and $3/\underline{60^\circ}$. Then the resultant is $\sqrt{4 + 9 + 12 \times \frac{1}{2}} / \underline{\tan^{-1} \frac{3\sqrt{3}}{7}} = 4.36/\underline{36^\circ 30'}$.

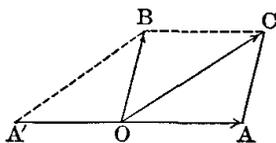
If the first component is given as $f_1/\underline{\theta_1}$, then we have the more symmetrical formula

$$OC = \sqrt{f_1^2 + f_2^2 + 2f_1f_2 \cos(\theta_2 - \theta_1)} / \underline{\tan^{-1} \frac{f_1 \sin \theta_1 + f_2 \sin \theta_2}{f_1 \cos \theta_1 + f_2 \cos \theta_2}}.$$

When the components are equal, the direction of the resultant bisects the angle formed by the vectors; and the magnitude of the resultant is twice the projection of either component on the bisecting line. The above formula reduces to

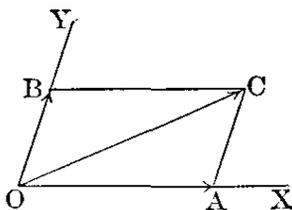
$$OC = 2f_1 \cos \frac{\theta_2}{2} / \underline{\frac{\theta_2}{2}}.$$

Example.—The resultant of two equal alternating electromotive forces which differ 120° in phase is equal in magnitude to either and has a phase of 60° .



Given a vector and one component, to find the other component.—Let OC represent the resultant, and OA the component. Join AC and draw OB equal and parallel to AC . The line OB represents the component required, for it is the only line which combined with OA gives OC as resultant. The line OB is identical with the diagonal of the parallelogram formed by OC and OA reversed; hence the rule is, “Reverse the direction of the component, then compound it with the given resultant to find the required component.” Let f/θ be the vector and f_1/θ_1 one component; then the other component is

$$f_2/\theta_2 = \sqrt{f^2 + f_1^2 - 2ff_1 \cos \theta} / \tan^{-1} \frac{f \sin \theta}{-f_1 + f \cos \theta}$$



Given the resultant and the directions of the two components, to find the magnitude of the components.—The resultant is represented by OC , and the directions by OX and OY . From C draw CA parallel to OY , and CB parallel to OX ; the lines OA and OB cut off represent the required components. It is evident that OA and OB when compounded produce the given resultant OC , and there is only one set of two components which produces a given resultant; hence they are the only pair of components having the given directions.

Let f/θ be the vector and f_1/θ_1 and f_2/θ_2 the given directions. Then

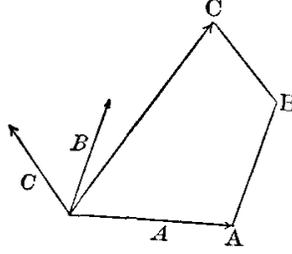
$$\begin{aligned} f_1 + f_2 \cos(\theta_2 - \theta_1) &= f \cos(\theta - \theta_1), \\ f_1 \cos(\theta_2 - \theta_1) + f_2 &= f \cos(\theta_2 - \theta), \end{aligned}$$

from which it follows that

$$f_1 = f \frac{\{\cos(\theta - \theta_1) - \cos(\theta_2 - \theta) \cos(\theta_2 - \theta_1)\}}{1 - \cos^2(\theta_2 - \theta_1)}.$$

For example, let $100/60^\circ$, $/30^\circ$, and $/90^\circ$ be given; then

$$f_1 = 100 \frac{\cos 30^\circ}{1 + \cos 60^\circ}.$$



Composition of any Number of Vectors applied at a common Point.—The resultant may be found by the following graphic construction: Take the vectors in any order, as A, B, C . From the end of A draw B' equal and parallel to B , and from the end of B' draw C' equal and parallel to C ; the vector from the beginning of A to the end of C' is the resultant of the given vectors. This follows by continued application of the parallelogram construction. The resultant obtained is the same, whatever the order; and as the order is arbitrary, the area enclosed has no physical meaning.

The result may be obtained analytically as follows:

Given

$$f_1/\underline{\theta}_1 + f_2/\underline{\theta}_2 + f_3/\underline{\theta}_3 + \cdots + f_n/\underline{\theta}_n.$$

Now

$$f_1/\underline{\theta}_1 = f_1 \cos \theta_1/\underline{0} + f_1 \sin \theta_1/\underline{\frac{\pi}{2}}.$$

Similarly

$$f_2/\underline{\theta}_2 = f_2 \cos \theta_2/\underline{0} + f_2 \sin \theta_2/\underline{\frac{\pi}{2}}.$$

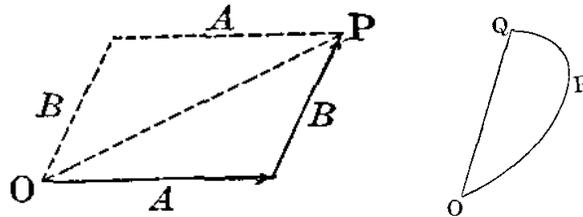
and

$$f_n/\underline{\theta}_n = f_n \cos \theta_n/\underline{0} + f_n \sin \theta_n/\underline{\frac{\pi}{2}}.$$

Hence

$$\begin{aligned} \sum \{f/\underline{\theta}\} &= \left\{ \sum f \cos \theta \right\} / \underline{0} + \left\{ \sum f \sin \theta \right\} / \underline{\frac{\pi}{2}} \\ &= \sqrt{\left(\sum f \cos \theta \right)^2 + \left(\sum f \sin \theta \right)^2} \cdot \tan^{-1} \frac{\sum f \sin \theta}{\sum f \cos \theta}. \end{aligned}$$

In the case of a sum of simultaneous vectors applied at a common point, the ordinary rule about the transposition of a term in an equation holds good. For example, if $A + B + C = 0$, then $A + B = -C$, and $A + C = -B$, and $B + C = -A$, etc. This is permissible because there is no real order of succession among the given components.³



Composition of Successive Vectors.—The composition of successive vectors partakes more of the nature of multiplication than of addition. Let A be a vector starting from the point O , and B a vector starting from the end of A . Draw the third side OP , and from O draw a vector equal to B , and from its extremity a vector equal to A . The line OP is not the complete equivalent of $A + B$; if it were so, it would also be the complete equivalent of $B + A$. But $A + B$ and $B + A$ determine different paths; and as they go oppositely around, the areas they determine with OP have different signs. The diagonal OP represents $A + B$ only so far as it is considered independent of path. For any number of successive vectors, the sum so far as it is independent of path is the vector from the initial point of the first to the final point of the last. This is also true when the successive vectors become so small as to form a continuous curve. The area between the curve OPQ and the vector OQ depends on the path, and has a physical meaning.

- Prob. 1. The resultant vector is $123/45^\circ$, and one component is $100/0^\circ$; find the other component.
- Prob. 2. The velocity of a body in a given plane is $200/75^\circ$, and one component is $100/25^\circ$; find the other component.
- Prob. 3. Three alternating magnetomotive forces are of equal virtual value, but each pair differs in phase by 120° ; find the resultant. (Ans. Zero.)
- Prob. 4. Find the components of the vector $100/70^\circ$ in the directions 20° and 100° .
- Prob. 5. Calculate the resultant vector of $1/10^\circ$, $2/20^\circ$, $3/30^\circ$, $4/40^\circ$.
- Prob. 6. Compound the following magnetic fluxes: $h \sin nt + h \sin(nt - 120^\circ)/120^\circ + h \sin(nt - 240^\circ)/240^\circ$. (Ans. $\frac{3}{2}h/nt$.)

³This does not hold true of a sum of vectors having a real order of succession. It is a mistake to attempt to found space-analysis upon arbitrary formal laws; the fundamental rules must be made to express universal properties of the thing denoted. In this chapter no attempt is made to apply formal laws to directed quantities. What is attempted is an analysis of these quantities.

- Prob. 7. Compound two alternating magnetic fluxes at a point $a \cos nt/0$ and $a \sin nt/\frac{\pi}{2}$.
(Ans. a/nt .)
- Prob. 8. Find the resultant of two simple alternating electromotive forces $100/20^\circ$ and $50/75^\circ$.
- Prob. 9. Prove that a uniform circular motion is obtained by compounding two equal simple harmonic motions which have the space-phase of their angular positions equal to the supplement of the time-phase of their motions.

Article 3

Products of Coplanar Vectors.

When all the vectors considered are confined to a common plane, each may be expressed as the sum of two rectangular components. Let i and j denote two directions in the plane at right angles to one another; then $A = a_1i + a_2j$, $B = b_1i + b_2j$, $R = xi + yj$. Here i and j are not unit-vectors, but rather signs of direction.

Product of two Vectors.—Let $A = a_1i + a_2j$ and $B = b_1i + b_2j$ be any two vectors, not necessarily of the same kind physically. We assume that their product is obtained by applying the distributive law, but we do not assume that the order of the factors is indifferent. Hence

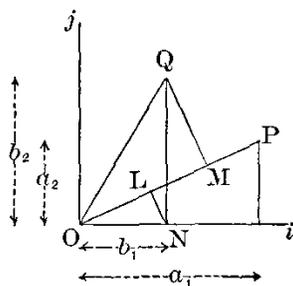
$$AB = (a_1i + a_2j)(b_1i + b_2j) = a_1b_1ii + a_2b_2jj + a_1b_2ij + a_2b_1ji.$$

If we assume, as suggested by ordinary algebra, that the square of a sign of direction is $+$, and further that the product of two directions at right angles to one another is the direction normal to both, then the above reduces to

$$AB = a_1b_1 + a_2b_2 + (a_1b_2 - a_2b_1)k.$$

Thus the complete product breaks up into two partial products, namely, $a_1b_1 + a_2b_2$ which is independent of direction, and $(a_1b_2 - a_2b_1)k$ which has the axis of the plane for direction.¹

¹A common explanation which is given of $ij = k$ is that i is an operator, j an operand, and k the result. The kind of operator which i is supposed to denote is a quadrant of turning round the axis i ; it is supposed not to be an axis, but a quadrant of rotation round an axis. This explains the result $ij = k$, but unfortunately it does not explain $ii = +$; for it would give $ii = i$.



Scalar Product of two Vectors.—By a scalar quantity is meant a quantity which has magnitude and may be positive or negative but is destitute of direction. The former partial product is so called because it is of such a nature. It is denoted by SAB where the symbol S , being in Roman type, denotes, not a vector, but a function of the vectors A and B . The geometrical meaning of SAB is the product of A and the orthogonal projection of B upon A . Let OP and OQ represent the vectors A and B ; draw QM and NL perpendicular to OP . Then

$$\begin{aligned} (OP)(OM) &= (OP)(OL) + (OP)(LM), \\ &= a \left\{ b_1 \frac{a_1}{a} + b_2 \frac{a_2}{a} \right\}, \\ &= a_1 b_1 + a_2 b_2. \end{aligned}$$

Corollary 1.— $SBA = SAB$. For instance, let A denote a force and B the velocity of its point of application; then SAB denotes the rate of working of the force. The result is the same whether the force is projected on the velocity or the velocity on the force.

Example 1.—A force of 2 pounds East + 3 pounds North is moved with a velocity of 4 feet East per second + 5 feet North per second; find the rate at which work is done.

$$2 \times 4 + 3 \times 5 = 23 \text{ foot-pounds per second.}$$

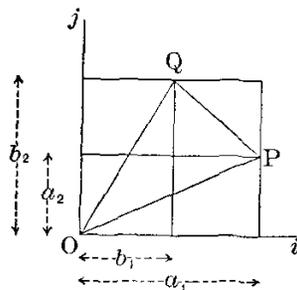
Corollary 2.— $A^2 = a_1^2 + a_2^2 = a^2$. The square of any vector is independent of direction; it is an essentially positive or signless quantity; for whatever the direction of A , the direction of the other A must be the same; hence the scalar product cannot be negative.

Example 2.—A stone of 10 pounds mass is moving with a velocity 64 feet down per second + 100 feet horizontal per second. Its kinetic energy then is

$$\frac{10}{2}(64^2 + 100^2) \text{ foot-poundals,}$$

a quantity which has no direction. The kinetic energy due to the downward velocity is $10 \times \frac{64^2}{2}$ and that due to the horizontal velocity is $\frac{10}{2} \times 100^2$; the

whole kinetic energy is obtained, not by vector, but by simple addition, when the components are rectangular.



Vector Product of two Vectors.—The other partial product from its nature is called the vector product, and is denoted by VAB . Its geometrical meaning is the product of A and the projection of B which is perpendicular to A , that is, the area of the parallelogram formed upon A and B . Let OP and OQ represent the vectors A and B , and draw the lines indicated by the figure. It is then evident that the area of the triangle $OPQ = a_1b_2 - \frac{1}{2}a_2a_2 - \frac{1}{2}b_1b_2 - \frac{1}{2}(a_1 - b_1)(b_2 - a_2) = \frac{1}{2}(a_1b_2 - a_2b_1)$.

Thus $(a_1b_2 - a_2b_1)k$ denotes the magnitude of the parallelogram formed by A and B and also the axis of the plane in which it lies.

It follows that $VBA = -VAB$. It is to be observed that the coordinates of A and B are mere component vectors, whereas A and B themselves are taken in a real order.

Example.—Let $A = (10i + 11j)$ inches and $B = (5i + 12j)$ inches, then $VAB = (120 - 55)k$ square inches; that is, 65 square inches in the plane which has the direction k for axis.

If A is expressed as $a\alpha$ and B as $b\beta$, then $SAB = ab \cos \alpha\beta$, where $\alpha\beta$ denotes the angle between the directions α and β .

Example.—The effective electromotive force of 100 volts per inch $/90^\circ$ along a conductor 8 inch $/45^\circ$ is $SAB = 8 \times 100 \cos /45^\circ/90^\circ$ volts, that is, $800 \cos 45^\circ$ volts. Here $/45^\circ$ indicates the direction α and $/90^\circ$ the direction β , and $/45^\circ/90^\circ$ means the angle between the direction of 45° and the direction of 90° .

Also $VAB = ab \sin \alpha\beta \cdot \overline{\alpha\beta}$, where $\overline{\alpha\beta}$ denotes the direction which is normal to both α and β , that is, their pole.

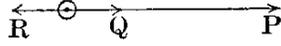
Example.—At a distance of 10 feet $/30^\circ$ there is a force of 100 pounds $/60^\circ$. The moment is VAB

$$\begin{aligned} &= 10 \times 100 \sin /30^\circ/60^\circ \text{ pound-feet } \overline{90^\circ}/90^\circ \\ &= 1000 \sin 30^\circ \text{ pound feet } \overline{90^\circ}/90^\circ \end{aligned}$$

Here $\overline{90^\circ}$ specifies the plane of the angle and $/90^\circ$ the angle. The two together written as above specify the normal k .

Reciprocal of a Vector.—By the reciprocal of a vector is meant the vector which combined with the original vector produces the product $+1$. The reciprocal of A is denoted by A^{-1} . Since $AB = ab(\cos \alpha\beta + \sin \alpha\beta \cdot \overline{\alpha\beta})$, b must equal a^{-1} and β must be identical with α in order that the product may be 1. It follows that

$$A^{-1} = \frac{1}{a}\alpha = \frac{a\alpha}{a^2} = \frac{a_1i + a_2j}{a_1^2 + a_2^2}.$$



The reciprocal and opposite vector is $-A^{-1}$. In the figure let $OP = 2\beta$ be the given vector; then $OQ = \frac{1}{2}\beta$ is its reciprocal, and $OR = \frac{1}{2}(-\beta)$ is its reciprocal and opposite.²

Example.—If $A = 10$ feet East + 5 feet North, $A^{-1} = \frac{10}{125}$ feet East + $\frac{5}{125}$ feet North and $-A^{-1} = -\frac{10}{125}$ feet East - $\frac{5}{125}$ feet North.

Product of the reciprocal of a vector and another vector.—

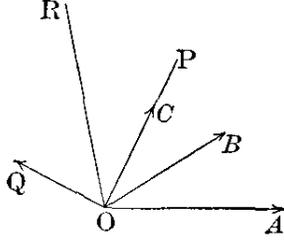
$$\begin{aligned} A^{-1}B &= \frac{1}{a^2}AB, \\ &= \frac{1}{a^2} \{a_1b_1 + a_2b_2 + (a_1b_2 - a_2b_1)\overline{\alpha\beta}\}, \\ &= \frac{b}{a}(\cos \alpha\beta + \sin \alpha\beta \cdot \overline{\alpha\beta}). \end{aligned}$$

Hence $SA^{-1}B = \frac{b}{a} \cos \alpha\beta$ and $VA^{-1}B = \frac{b}{a} \sin \alpha\beta \cdot \overline{\alpha\beta}$.

²Writers who identify a vector with a quadrantal versor are logically led to define the reciprocal of a vector as being opposite in direction as well as reciprocal in magnitude.

Product of three Coplanar Vectors.—Let $A = a_1i + a_2j$, $B = b_1i + b_2j$, $C = c_1i + c_2j$ denote any three vectors in a common plane. Then

$$\begin{aligned}(AB)C &= \{(a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)k\}(c_1i + c_2j) \\ &= (a_1b_1 + a_2b_2)(c_1i + c_2j) + (a_1b_2 - a_2b_1)(-c_2i + c_1j).\end{aligned}$$



The former partial product means the vector C multiplied by the scalar product of A and B ; while the latter partial product means the complementary vector of C multiplied by the magnitude of the vector product of A and B . If these partial products (represented by OP and OQ) unite to form a total product, the total product will be represented by OR , the resultant of OP and OQ .

The former product is also expressed by $SAB \cdot C$, where the point separates the vectors to which the S refers; and more analytically by $abc \cos \alpha\beta \cdot \gamma$.

The latter product is also expressed by $(VAB)C$, which is equivalent to $V(VAB)C$, because VAB is at right angles to C . It is also expressed by $abc \sin \alpha\beta \cdot \overline{\alpha\beta\gamma}$, where $\overline{\alpha\beta\gamma}$ denotes the direction which is perpendicular to the perpendicular to α and β and γ .

If the product is formed after the other mode of association we have

$$\begin{aligned}A(BC) &= (a_1i + a_2j)(b_1c_1 + b_2c_2) + (a_1i + a_2j)(b_1c_2 - b_2c_1)k \\ &= (b_1c_1 + b_2c_2)(a_1i + a_2j) + (b_1c_2 - b_2c_1)(a_2i - a_1j) \\ &= SBC \cdot A + VA(VBC).\end{aligned}$$

The vector $a_2i - a_1j$ is the opposite of the complementary vector of $a_1i + a_2j$. Hence the latter partial product differs with the mode of association.

Example.—Let $A = 1/0^\circ + 2/90^\circ$, $B = 3/0^\circ + 4/90^\circ$, $C = 5/0^\circ + 6/90^\circ$. The fourth proportional to A, B, C is

$$\begin{aligned}(A^{-1}B)C &= \frac{1 \times 3 + 2 \times 4}{1^2 + 2^2} \{5/0^\circ + 6/90^\circ\} \\ &\quad + \frac{1 \times 4 - 2 \times 3}{1^2 + 2^2} \{-6/0^\circ + 5/90^\circ\} \\ &= 13.4/0^\circ + 11.2/90^\circ.\end{aligned}$$

Square of a Binomial of Vectors.—If $A + B$ denotes a sum of non-successive vectors, it is entirely equivalent to the resultant vector C . But the square of

any vector is a positive scalar, hence the square of $A + B$ must be a positive scalar. Since A and B are in reality components of one vector, the square must be formed after the rules for the products of rectangular components (p. 432). Hence

$$\begin{aligned}(A + B)^2 &= (A + B)(A + B), \\ &= A^2 + AB + BA + B^2, \\ &= A^2 + B^2 + SAB + SBA + VAB + VBA, \\ &= A^2 + B^2 + 2SAB.\end{aligned}$$

This may also be written in the form

$$a^2 + b^2 + 2ab \cos \alpha\beta.$$

But when $A + B$ denotes a sum of successive vectors, there is no third vector C which is the complete equivalent; and consequently we need not expect the square to be a scalar quantity. We observe that there is a real order, not of the factors, but of the terms in the binomial; this causes both product terms to be AB , giving

$$\begin{aligned}(A + B)^2 &= A^2 + 2AB + B^2 \\ &= A^2 + B^2 + 2SAB + 2VAB.\end{aligned}$$

The scalar part gives the square of the length of the third side, while the vector part gives four times the area included between the path and the third side.

Square of a Trinomial of Coplanar Vectors.—Let $A + B + C$ denote a sum of successive vectors. The product terms must be formed so as to preserve the order of the vectors in the trinomial; that is, A is prior to B and C , and B is prior to C . Hence

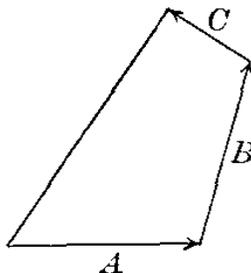
$$\begin{aligned}(A + B + C)^2 &= A^2 + B^2 + C^2 + 2AB + 2AC + 2BC, \\ &= A^2 + B^2 + C^2 + 2(SAB + SAC + SBC), \quad (1) \\ &\quad + 2(VAB + VAC + VBC). \quad (2)\end{aligned}$$

Hence

$$\begin{aligned}S(A + B + C)^2 &= (1) \\ &= a^2 + b^2 + c^2 + 2ab \cos \alpha\beta + 2ac \cos \alpha\gamma + 2bc \cos \beta\gamma\end{aligned}$$

and

$$\begin{aligned}V(A + B + C)^2 &= (2) \\ &= \{2ab \sin \alpha\beta + 2ac \sin \alpha\gamma + 2bc \sin \beta\gamma\} \cdot \overline{\alpha\beta}\end{aligned}$$



The scalar part gives the square of the vector from the beginning of A to the end of C and is all that exists when the vectors are non-successive. The vector part is four times the area included between the successive sides and the resultant side of the polygon.

Note that it is here assumed that $V(A + B)C = VAC + VBC$, which is the theorem of moments. Also that the product terms are not formed in cyclical order, but in accordance with the order of the vectors in the trinomial.

Example.—Let $A = 3/0^\circ$, $B = 5/30^\circ$, $C = 7/45^\circ$; find the area of the polygon.

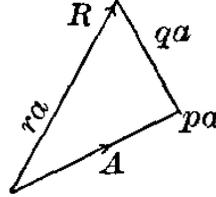
$$\begin{aligned} \frac{1}{2}V(AB + AC + BC) &= \frac{1}{2}\{15 \sin /0^\circ/30^\circ + 21 \sin /0^\circ/45^\circ + 35/30^\circ/45^\circ\}, \\ &= 3.75 + 7.42 + 4.53 = 15.7. \end{aligned}$$

- Prob. 10. At a distance of 25 centimeters $/20^\circ$ there is a force of 1000 dynes $/80^\circ$; find the moment.
- Prob. 11. A conductor in an armature has a velocity of 240 inches per second $/300^\circ$ and the magnetic flux is 50,000 lines per square inch $/0^\circ$; find the vector product. (Ans. 1.04×10^7 lines per inch per second.)
- Prob. 12. Find the sine and cosine of the angle between the directions $0.8141 \text{ E.} + 0.5807 \text{ N.}$, and $0.5060 \text{ E.} + 0.8625 \text{ N.}$
- Prob. 13. When a force of 200 pounds $/270^\circ$ is displaced by 10 feet $/30^\circ$, what is the work done (scalar product)? What is the meaning of the negative sign in the scalar product?
- Prob. 14. A mass of 100 pounds is moving with a velocity of 30 feet E. per second + 50 feet SE. per second; find its kinetic energy.
- Prob. 15. A force of 10 pounds $/45^\circ$ is acting at the end of 8 feet $/200^\circ$; find the torque, or vector product.
- Prob. 16. The radius of curvature of a curve is $2/0^\circ + 5/90^\circ$; find the curvature. (Ans. $.03/0^\circ + .17/90^\circ$.)
- Prob. 17. Find the fourth proportional to $10/0^\circ + 2/90^\circ$, $8/0^\circ - 3/90^\circ$, and $6/0^\circ + 5/90^\circ$.
- Prob. 18. Find the area of the polygon whose successive sides are $10/30^\circ$, $9/100^\circ$, $8/180^\circ$, $7/225^\circ$.

Article 4

Coaxial Quaternions.

By a “quaternion” is meant the operator which changes one vector into another. It is composed of a magnitude and a turning factor. The magnitude may or may not be a mere ratio, that is, a quantity destitute of physical dimensions; for the two vectors may or may not be of the same physical kind. The turning is in a plane, that is to say, it is not conical. For the present all the vectors considered lie in a common plane; hence all the quaternions considered have a common axis.¹



Let A and R be two coinitial vectors; the direction normal to the plane may be denoted by β . The operator which changes A into R consists of a scalar multiplier and a turning round the axis β . Let the former be denoted by r and the latter by β^θ , where θ denotes the angle in radians. Thus $R = r\beta^\theta A$ and reciprocally $A = \frac{1}{r}\beta^{-\theta}R$. Also $\frac{1}{A}R = r\beta^\theta$ and $\frac{1}{R}A = \frac{1}{r}\beta^{-\theta}$.

The turning factor β^θ may be expressed as the sum of two component operators, one of which has a zero angle and the other an angle of a quadrant. Thus

$$\beta^\theta = \cos \theta \cdot \beta^\theta + \sin \theta \cdot \beta^{\frac{\pi}{2}}.$$

When the angle is naught, the turning-factor may be omitted; but the above form shows that the equation is homogeneous, and expresses nothing but the

¹The idea of the “quaternion” is due to Hamilton. Its importance may be judged from the fact that it has made solid trigonometrical analysis possible. It is the most important key to the extension of analysis to space. Etymologically “quaternion” means defined by four elements; which is true in space; in plane analysis it is defined by two.

equivalence of a given quaternion to two component quaternions.²

Hence

$$\begin{aligned} r\beta^\theta &= r \cos \theta + r \sin \theta \cdot \beta^{\frac{\pi}{2}} \\ &= p + q \cdot \beta^{\frac{\pi}{2}} \end{aligned}$$

and

$$\begin{aligned} r\beta^\theta A &= pA + q\beta^{\frac{\pi}{2}} A \\ &= pa \cdot \alpha + qa \cdot \beta^{\frac{\pi}{2}} \alpha. \end{aligned}$$

The relations between r and θ , and p and q , are given by

$$r = \sqrt{p^2 + q^2}, \quad \theta = \tan^{-1} \frac{p}{q}.$$

Example.—Let E denote a sine alternating electromotive force in magnitude and phase, and I the alternating current in magnitude and phase, then

$$E = (r + 2\pi nl \cdot \beta^{\frac{\pi}{2}}) I,$$

where r is the resistance, l the self-induction, n the alternations per unit of time, and β denotes the axis of the plane of representation. It follows that $E = rI + 2\pi nl \cdot \beta^{\frac{\pi}{2}} I$; also that

$$I^{-1}E = r + 2\pi nl \cdot \beta^{\frac{\pi}{2}},$$

that is, the operator which changes the current into the electromotive force is a quaternion. The resistance is the scalar part of the quaternion, and the inductance is the vector part.

Components of the Reciprocal of a Quaternion.—Given

$$R = (p + q \cdot \beta^{\frac{\pi}{2}}) A,$$

then

$$\begin{aligned} A &= \frac{1}{p + q \cdot \beta^{\frac{\pi}{2}}} R \\ &= \frac{p - q \cdot \beta^{\frac{\pi}{2}}}{(p + q \cdot \beta^{\frac{\pi}{2}})(p - q \cdot \beta^{\frac{\pi}{2}})} R \\ &= \frac{p - q \cdot \beta^{\frac{\pi}{2}}}{p^2 + q^2} R \\ &= \left\{ \frac{p}{p^2 + q^2} - \frac{q}{p^2 + q^2} \cdot \beta^{\frac{\pi}{2}} \right\} R. \end{aligned}$$

²In the method of complex numbers $\beta^{\frac{\pi}{2}}$ is expressed by i , which stands for $\sqrt{-1}$. The advantages of using the above notation are that it is capable of being applied to space, and that it also serves to specify the general turning factor β^θ as well as the quadrantal turning factor $\beta^{\frac{\pi}{2}}$.

Example.—Take the same application as above. It is important to obtain I in terms of E . By the above we deduce that from $E = (r + 2\pi nl \cdot \beta^{\frac{\pi}{2}})I$

$$I = \left\{ \frac{r}{r^2 + (2\pi nl)^2} - \frac{2\pi nl}{r^2 + (2\pi nl)^2} \cdot \beta^{\frac{\pi}{2}} \right\} E.$$

Addition of Coaxial Quaternions.—If the ratio of each of several vectors to a constant vector A is given, the ratio of their resultant to the same constant vector is obtained by taking the sum of the ratios. Thus, if

$$\begin{aligned} R_1 &= (p_1 + q_1 \cdot \beta^{\frac{\pi}{2}})A, \\ R_2 &= (p_2 + q_2 \cdot \beta^{\frac{\pi}{2}})A, \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \\ R_n &= (p_n + q_n \cdot \beta^{\frac{\pi}{2}})A, \end{aligned}$$

then

$$\sum R = \left\{ \sum p + \left(\sum q \right) \cdot \beta^{\frac{\pi}{2}} \right\} A,$$

and reciprocally

$$A = \frac{\sum p - \left(\sum q \right) \cdot \beta^{\frac{\pi}{2}}}{\left(\sum p \right)^2 + \left(\sum q \right)^2} \sum R.$$

Example.—In the case of a compound circuit composed of a number of simple circuits in parallel

$$I_1 = \frac{r_1 - 2\pi nl_1 \cdot \beta^{\frac{\pi}{2}}}{r_1^2 + (2\pi n)^2 l_1^2} E, \quad I_2 = \frac{r_2 - 2\pi nl_2 \cdot \beta^{\frac{\pi}{2}}}{r_2^2 + (2\pi n)^2 l_2^2} E, \quad \text{etc.},$$

therefore,

$$\begin{aligned} \sum I &= \sum \left\{ \frac{r - 2\pi nl \cdot \beta^{\frac{\pi}{2}}}{r^2 + (2\pi n)^2 l^2} \right\} E \\ &= \left\{ \sum \left(\frac{r}{r^2 + (2\pi n)^2 l^2} \right) - 2\pi n \sum \frac{l}{r^2 + (2\pi n)^2 l^2} \cdot \beta^{\frac{\pi}{2}} \right\} E, \end{aligned}$$

and reciprocally

$$E = \frac{\sum \left(\frac{r}{r^2 + (2\pi n)^2 l^2} \right) + 2\pi n \sum \left(\frac{l}{r^2 + (2\pi n)^2 l^2} \right) \cdot \beta^{\frac{\pi}{2}}}{\left(\sum \frac{r}{r^2 + (2\pi n)^2 l^2} \right)^2 + (2\pi n)^2 \left(\sum \frac{l}{r^2 + (2\pi n)^2 l^2} \right)^2} \sum I.^3$$

³This theorem was discovered by Lord Rayleigh; *Philosophical Magazine*, May, 1886. See also Bedell & Crehore's *Alternating Currents*, p. 238.

Product of Coaxial Quaternions.—If the quaternions which change A to R , and R to R' , are given, the quaternion which changes A to R' is obtained by taking the product of the given quaternions.

Given

$$R = r\beta^\theta A = (p + q \cdot \beta^{\frac{\pi}{2}}) A$$

and

$$R' = r'\beta^{\theta'} A = (p' + q' \cdot \beta^{\frac{\pi}{2}}) R,$$

then

$$R' = rr'\beta^{\theta+\theta'} A = \{(pp' - qq') + (pq' + p'q) \cdot \beta^{\frac{\pi}{2}}\} A.$$

Note that the product is formed by taking the product of the magnitudes, and likewise the product of the turning factors. The angles are summed because they are indices of the common base β .⁴

Quotient of two Coaxial Quaternions.—If the given quaternions are those which change A to R , and A to R' , then that which changes R to R' is obtained by taking the quotient of the latter by the former.

Given

$$R = r\beta^\theta A = (p + q \cdot \beta^{\frac{\pi}{2}}) A$$

and

$$R' = r'\beta^{\theta'} A = (p' + q' \cdot \beta^{\frac{\pi}{2}}) A,$$

then

$$\begin{aligned} R' &= \frac{r'}{r} \beta^{\theta' - \theta} R, \\ &= (p' + q' \cdot \beta^{\frac{\pi}{2}}) \frac{1}{p + q \cdot \beta^{\frac{\pi}{2}}} R, \\ &= (p' + q' \cdot \beta^{\frac{\pi}{2}}) \frac{p - q \cdot \beta^{\frac{\pi}{2}}}{p^2 + q^2} R, \\ &= \frac{(pp' + qq') + (pq' - p'q) \cdot \beta^{\frac{\pi}{2}}}{p^2 + q^2} R. \end{aligned}$$

- Prob. 19. The impressed alternating electromotive force is 200 volts, the resistance of the circuit is 10 ohms, the self-induction is $\frac{1}{100}$ henry, and there are 60 alternations per second; required the current. (Ans. 18.7 amperes / $-20^\circ 42'$.)

⁴Many writers, such as Hayward in "Vector Algebra and Trigonometry," and Stringham in "Uniplanar Algebra," treat this product of coaxial quaternions as if it were the product of vectors. This is the fundamental error in the Argand method.

- Prob. 20. If in the above circuit the current is 10 amperes, find the impressed voltage.
- Prob. 21. If the electromotive force is 110 volts $\sqrt{\theta}$ and the current is 10 amperes $\sqrt{\theta - \frac{1}{4}\pi}$, find the resistance and the self-induction, there being 120 alternations per second.
- Prob. 22. A number of coils having resistances $r_1, r_2, \text{ etc.}$, and self-inductions $l_1, l_2, \text{ etc.}$, are placed in series; find the impressed electromotive force in terms of the current, and reciprocally.

Article 5

Addition of Vectors in Space.

A vector in space can be expressed in terms of three independent components, and when these form a rectangular set the directions of resolution are expressed by i, j, k . Any variable vector R may be expressed as $R = r\rho = xi + yj + zk$, and any constant vector B may be expressed as

$$B = b\beta = b_1i + b_2j + b_3k.$$

In space the symbol ρ for the direction involves two elements. It may be specified as

$$\rho = \frac{xi + yj + zk}{x^2 + y^2 + z^2},$$

where the three squares are subject to the condition that their sum is unity. Or it may be specified by this notation, $\overline{\phi}/\underline{\theta}$, a generalization of the notation for a plane. The additional angle $\overline{\phi}$ is introduced to specify the plane in which the angle from the initial line lies.

If we are given R in the form $r\overline{\phi}/\underline{\theta}$, then we deduce the other form thus:

$$R = r \cos \theta \cdot i + r \sin \theta \cos \phi \cdot j + r \sin \theta \sin \phi \cdot k.$$

If R is given in the form $xi + yj + zk$, we deduce

$$R = \sqrt{x^2 + y^2 + z^2} \overline{\tan^{-1} \frac{z}{y}} / \underline{\tan^{-1} \frac{\sqrt{y^2 + z^2}}{x}}.$$

For example,

$$\begin{aligned} B &= 10 \overline{30^\circ} / \underline{45^\circ} \\ &= 10 \cos 45^\circ \cdot i + 10 \sin 45^\circ \cos 30^\circ \cdot j + 10 \sin 45^\circ \sin 30^\circ \cdot k. \end{aligned}$$

Again, from $C = 3i + 4j + 5k$ we deduce

$$C = \sqrt{9 + 16 + 25} \overline{\tan^{-1} \frac{5}{4}} // \overline{\tan^{-1} \frac{\sqrt{41}}{3}}$$

$$= 7.07 \overline{51^\circ.4} // \overline{64^\circ.9}.$$

To find the resultant of any number of component vectors applied at a common point, let R_1, R_2, \dots, R_n represent the n vectors or,

$$R_1 = x_1i + y_1j + z_1k,$$

$$R_2 = x_2i + y_2j + z_2k,$$

.....

$$R_n = x_ni + y_nj + z_nk;$$

then

$$\sum R = \left(\sum x\right)i + \left(\sum y\right)j + \left(\sum z\right)k$$

and

$$r = \sqrt{\left(\sum x\right)^2 + \left(\sum y\right)^2 + \left(\sum z\right)^2},$$

$$\tan \phi = \frac{\sum z}{\sum y} \text{ and } \tan \theta = \frac{\sqrt{\left(\sum y\right)^2 + \left(\sum z\right)^2}}{\sum x}.$$

Successive Addition.—When the successive vectors do not lie in one plane, the several elements of the area enclosed will lie in different planes, but these add by vector addition into a resultant directed area.

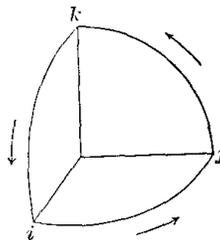
- Prob. 23. Express $A = 4i - 5j + 6k$ and $B = 5i + 6j - 7k$ in the form $r\overline{\phi} // \overline{\theta}$
 (Ans. $8.8 \overline{130^\circ} // \overline{63^\circ}$ and $10.5 \overline{311^\circ} // \overline{61^\circ.5}$.)
- Prob. 24. Express $C = 123 \overline{57^\circ} // \overline{142^\circ}$ and $D = 456 \overline{65^\circ} // \overline{200^\circ}$ in the form $xi + yj + zk$.
- Prob. 25. Express $E = 100 \overline{\frac{\pi}{4}} // \overline{\frac{\pi}{3}}$ and $F = 1000 \overline{\frac{\pi}{6}} // \overline{\frac{3\pi}{4}}$ in the form $xi + yj + zk$.
- Prob. 26. Find the resultant of $10 \overline{20^\circ} // \overline{30^\circ}$, $20 \overline{30^\circ} // \overline{40^\circ}$, and $30 \overline{40^\circ} // \overline{50^\circ}$.
- Prob. 27. Express in the form $r\overline{\phi} // \overline{\theta}$ the resultant vector of $1i + 2j - 3k$, $4i - 5j + 6k$ and $-7i + 8j + 9k$.

Article 6

Product of Two Vectors.

Rules of Signs for Vectors in Space.—By the rules $i^2 = +$, $j^2 = +$, $ij = k$, and $ji = -k$ we obtained (p. 432) a product of two vectors containing two partial products, each of which has the highest importance in mathematical and physical analysis. Accordingly, from the symmetry of space we assume that the following rules are true for the product of two vectors in space:

$$\begin{array}{lll} i^2 = +, & j^2 = +, & k^2 = +, \\ ij = k, & jk = i, & ki = j, \\ ji = -k, & kj = -i, & ik = -j. \end{array}$$



The square combinations give results which are independent of direction, and consequently are summed by simple addition. The area vector determined by i and j can be represented in direction by k , because k is in tri-dimensional space the axis which is complementary to i and j . We also observe that the three rules $ij = k$, $jk = i$, $ki = j$ are derived from one another by cyclical permutation; likewise the three rules $ji = -k$, $kj = -i$, $ik = -j$. The figure shows that these rules are made to represent the relation of the advance to the rotation in the right-handed screw. The physical meaning of these rules is made clearer by an application to the dynamo and the electric motor. In the dynamo three principal vectors have to be considered: the velocity of the conductor at any instant, the intensity of magnetic flux, and the vector of electromotive force. Frequently all that is demanded is, given two of these directions to determine

the third. Suppose that the direction of the velocity is i , and that of the flux j , then the direction of the electromotive force is k . The formula $ij = k$ becomes

$$\text{velocity flux} = \text{electromotive-force},$$

from which we deduce

$$\text{flux electromotive-force} = \text{velocity},$$

and

$$\text{electromotive-force velocity} = \text{flux}.$$

The corresponding formula for the electric motor is

$$\text{current flux} = \text{mechanical-force},$$

from which we derive by cyclical permutation

$$\text{flux force} = \text{current}, \quad \text{and} \quad \text{force current} = \text{flux}.$$

The formula $\text{velocity flux} = \text{electromotive-force}$ is much handier than any thumb-and-finger rule; for it compares the three directions directly with the right-handed screw.

Example.—Suppose that the conductor is normal to the plane of the paper, that its velocity is towards the bottom, and that the magnetic flux is towards the left; corresponding to the rotation from the velocity to the flux in the right-handed screw we have advance into the paper: that then is the direction of the electromotive force.

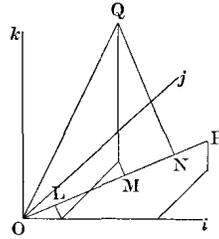
Again, suppose that in a motor the direction of the current along the conductor is up from the paper, and that the magnetic flux is to the left; corresponding to current flux we have advance towards the bottom of the page, which therefore must be the direction of the mechanical force which is applied to the conductor.

Complete Product of two Vectors.—Let $A = a_1i + a_2j + a_3k$ and $B = b_1i + b_2j + b_3k$ be any two vectors, not necessarily of the same kind physically, Their product, according to the rules (p. 444), is

$$\begin{aligned} AB &= (a_1i + a_2j + a_3k)(b_1i + b_2j + b_3k), \\ &= a_1b_1ii + a_2b_2jj + a_3b_3kk \\ &\quad + a_2b_3jk + a_3b_2kj + a_3b_1ki + a_1b_3ik + a_1b_2ij + a_2b_1ji \\ &= a_1b_1 + a_2b_2 + a_3b_3 \\ &\quad + (a_2b_3)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k \\ &= a_1b_1 + a_2b_2 + a_3b_3 + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ i & j & k \end{vmatrix} \end{aligned}$$

Thus the product breaks up into two partial products, namely, $a_1b_1 + a_2b_2 + a_3b_3$, which is independent of direction, and $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ i & j & k \end{vmatrix}$, which has the direction normal to the plane of A and B . The former is called the scalar product, and the latter the vector product.

In a sum of vectors, the vectors are necessarily homogeneous, but in a product the vectors may be heterogeneous. By making $a_3 = b_3 = 0$, we deduce the results already obtained for a plane.



Scalar Product of two Vectors.—The scalar product is denoted as before by SAB . Its geometrical meaning is the product of A and the orthogonal projection of B upon A . Let OP represent A , and OQ represent B , and let OL , LM , and MN be the orthogonal projections upon OP of the coordinates b_1i , b_2j , b_3k respectively. Then ON is the orthogonal projection of OQ , and

$$\begin{aligned} OP \times ON &= OP \times (OL + LM + MN), \\ &= a \left(b_1 \frac{a_1}{a} + b_2 \frac{a_2}{a} + b_3 \frac{a_3}{a} \right), \\ &= a_1b_1 + a_2b_2 + a_3b_3 = SAB. \end{aligned}$$

Example.—Let the intensity of a magnetic flux be $B = b_1i + b_2j + b_3k$, and let the area be $S = s_1i + s_2j + s_3k$; then the flux through the area is $SSB = b_1s_1 + b_2s_2 + b_3s_3$.

Corollary 1.—Hence $SBA = SAB$. For

$$b_1a_1 + b_2a_2 + b_3a_3 = a_1b_1 + a_2b_2 + a_3b_3.$$

The product of B and the orthogonal projection on it of A is equal to the product of A and the orthogonal projection on it of B . The product is positive when the vector and the projection have the same direction, and negative when they have opposite directions.

Corollary 2.—Hence $A^2 = a_1^2 + a_2^2 + a_3^2 = a^2$. The square of A must be positive; for the two factors have the same direction.

Vector Product of two Vectors.—The vector product as before is denoted by VAB . It means the product of A and the component of B which is perpendicular

to A , and is represented by the area of the parallelogram formed by A and B . The orthogonal projections of this area upon the planes of jk , ki , and ij represent the respective components of the product. For, let OP and OQ (see second figure of Art. 3) be the orthogonal projections of A and B on the plane of i and j ; then the triangle OPQ is the projection of half of the parallelogram formed by A and B . But it is there shown that the area of the triangle OPQ is $\frac{1}{2}(a_1b_2 - a_2b_1)$. Thus $(a_1b_2 - a_2b_1)k$ denotes the magnitude and direction of the parallelogram formed by the projections of A and B on the plane of i and j . Similarly $(a_2b_3 - a_3b_2)i$ denotes in magnitude and direction the projection on the plane of j and k , and $(a_3b_1 - a_1b_3)j$ that on the plane of k and i .

Corollary 1.—Hence $VBA = -VAB$.

Example.—Given two lines $A = 7i - 10j + 3k$ and $B = -9i + 4j - 6k$; to find the rectangular projections of the parallelogram which they define:

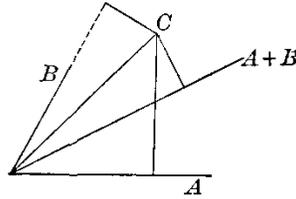
$$\begin{aligned} VAB &= (60 - 12)i + (-27 + 42)j + (28 - 90)k \\ &= 48i + 15j - 62k. \end{aligned}$$

Corollary 2.—If A is expressed as $a\alpha$ and B as $b\beta$, then $SAB = ab \cos \alpha\beta$ and $VAB = ab \sin \alpha\beta \cdot \overline{\alpha\beta}$, where $\overline{\alpha\beta}$ denotes the direction which is normal to both α and β , and drawn in the sense given by the right-handed screw.

Example.—Given $A = r \overline{\phi} / \theta$ and $B = r' \overline{\phi'} / \theta'$. Then

$$\begin{aligned} SAB &= rr' \cos \overline{\phi} / \theta \overline{\phi'} / \theta' \\ &= rr' \{ \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi' - \phi) \}. \end{aligned}$$

Product of two Sums of non-successive Vectors.—Let A and B be two component vectors, giving the resultant $A + B$, and let C denote any other vector having the same point of application.



Let

$$\begin{aligned} A &= a_1j + a_2j + a_3k, \\ B &= b_1i + b_2j + b_3k, \\ C &= c_1i + c_2j + c_3k. \end{aligned}$$

Since A and B are independent of order,

$$A + B = (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k,$$

consequently by the principle already established

$$\begin{aligned} S(A + B)C &= (a_1 + b_1)c_1 + (a_2 + b_2)c_2 + (a_3 + b_3)c_3 \\ &= a_1c_1 + a_2c_2 + a_3c_3 + b_1c_1 + b_2c_2 + b_3c_3 \\ &= SAC + SBC. \end{aligned}$$

Similarly

$$\begin{aligned} V(A + B)C &= \{(a_2 + b_2)c_3 - (a_3 + b_3)c_2\}i + \text{etc.} \\ &= (a_2c_3 - a_3c_2)i + (b_2c_3 - b_3c_2)i + \cdots \\ &= VAC + VBC. \end{aligned}$$

Hence $(A + B)C = AC + BC$.

In the same way it may be shown that if the second factor consists of two components, C and D , which are non-successive in their nature, then

$$(A + B)(C + D) = AC + AD + BC + BD.$$

When $A + B$ is a sum of component vectors

$$\begin{aligned} (A + B)^2 &= A^2 + B^2 + AB + BA \\ &= A^2 + B^2 + 2SAB. \end{aligned}$$

- Prob. 28. The relative velocity of a conductor is S.W., and the magnetic flux is N.W.; what is the direction of the electromotive force in the conductor?
- Prob. 29. The direction of the current is vertically downward, that of the magnetic flux is West; find the direction of the mechanical force on the conductor.
- Prob. 30. A body to which a force of $2i + 3j + 4k$ pounds is applied moves with a velocity of $5i + 6j + 7k$ feet per second; find the rate at which work is done.
- Prob. 31. A conductor $8i + 9j + 10k$ inches long is subject to an electromotive force of $11i + 12j + 13k$ volts per inch; find the difference of potential at the ends. (Ans. 326 volts.)
- Prob. 32. Find the rectangular projections of the area of the parallelogram defined by the vectors $A = 12i - 23j - 34k$ and $B = -45i - 56j + 67k$.
- Prob. 33. Show that the moment of the velocity of a body with respect to a point is equal to the sum of the moments of its component velocities with respect to the same point.
- Prob. 34. The arm is $9i + 11j + 13k$ feet, and the force applied at either end is $17i + 19j + 23k$ pounds weight; find the torque.
- Prob. 35. A body of 1000 pounds mass has linear velocities of 50 feet per second $\overline{30^\circ}/45^\circ$ and 60 feet per second $\overline{60^\circ}/22^\circ.5$; find its kinetic energy.
- Prob. 36. Show that if a system of area-vectors can be represented by the faces of a polyhedron, their resultant vanishes.
- Prob. 37. Show that work done by the resultant velocity is equal to the sum of the works done by its components.

Article 7

Product of Three Vectors.

Complete Product.—Let us take $A = a_1i + a_2j + a_3k$, $B = b_1i + b_2j + b_3k$, and $C = c_1i + c_2j + c_3k$. By the product of A , B , and C is meant the product of the product of A and B with C , according to the rules p. 444). Hence

$$\begin{aligned} ABC &= (a_1b_1 + a_2b_2 + a_3b_3)(c_1i + c_2j + c_3k) \\ &+ \left\{ (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k \right\} (c_1i + c_2j + c_3k) \\ &= (a_1b_1 + a_2b_2 + a_3b_3)(c_1i + c_2j + c_3k) \end{aligned} \quad (1)$$

$$+ \begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ c_1 & c_2 & c_3 \\ i & j & k \end{vmatrix} \quad (2)$$

$$+ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (3)$$

Example.—Let $A = i + 2j + 3k$, $B = 4i + 5j + 6k$, and $C = 7i + 8j + 9k$. Then

$$(1) = (4 + 10 + 18)(7i + 8j + 9k) = 32(7i + 8j + 9k).$$

$$(2) = \begin{vmatrix} -3 & 6 & -3 \\ 7 & 8 & 9 \\ i & j & k \end{vmatrix} = 78i + 6j - 66k.$$

$$(3) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0.$$

If we write $A = a\alpha$, $B = b\beta$, $C = c\gamma$, then

$$ABC = abc \cos \alpha\beta \cdot \gamma \quad (1)$$

$$+ abc \sin \alpha\beta \sin \overline{\alpha\beta\gamma} \cdot \overline{\alpha\beta\gamma} \quad (2)$$

$$+ abc \sin \alpha\beta \cos \overline{\alpha\beta}\gamma, \quad (3)$$

where $\cos \overline{\alpha\beta}\gamma$ denotes the cosine of the angle between the directions $\overline{\alpha\beta}$ and γ , and $\overline{\alpha\beta\gamma}$ denotes the direction which is normal to both $\overline{\alpha\beta}$ and γ .

We may also write

$$ABC = \underset{(1)}{SAB} \cdot C + \underset{(2)}{V(VAB)C} + \underset{(3)}{S(VAB)C}$$

First Partial Product.—It is merely the third vector multiplied by the scalar product of the other two, or weighted by that product as an ordinary algebraic quantity. If the directions are kept constant, each of the three partial products is proportional to each of the three magnitudes.

Second Partial Product.—The second partial product may be expressed as the difference of two products similar to the first. For

$$\begin{aligned} V(VAB)C &= \{-(b_2c_2 + b_3c_3)a_1 + (c_2a_2 + c_3a_3)b_1\}i \\ &+ \{-(b_3c_3 + b_1c_1)a_2 + (c_3a_3 + c_1a_1)b_2\}j \\ &+ \{-(b_1c_1 + b_2c_2)a_3 + (c_1a_1 + c_2a_2)b_3\}k. \end{aligned}$$

By adding to the first of these components the null term $(b_1c_1a_1 - c_1a_1b_1)i$ we get $-SBC \cdot a_1i + SCA \cdot b_1i$, and by treating the other two components similarly and adding the results we obtain

$$V(VAB)C = -SBC \cdot A + SCA \cdot B.$$

The principle here proved is of great use in solving equations (see p. 455).

Example.—Take the same three vectors as in the preceding example. Then

$$\begin{aligned} V(VAB)C &= -(28 + 40 + 54)(1i + 2j + 3k) \\ &+ (7 + 16 + 27)(4i + 5j + 6k) \\ &= 78i + 6j - 66k. \end{aligned}$$

The determinant expression for this partial product may also be written in the form

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} c_1 & c_2 \\ i & j \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \begin{vmatrix} c_2 & c_3 \\ j & k \end{vmatrix} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} \begin{vmatrix} c_3 & c_1 \\ k & i \end{vmatrix}$$

It follows that the frequently occurring determinant expression

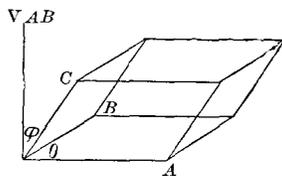
$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \begin{vmatrix} c_2 & c_3 \\ d_2 & d_3 \end{vmatrix} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} \begin{vmatrix} c_3 & c_1 \\ d_3 & d_1 \end{vmatrix}$$

means $S(VAB)(VCD)$.

Third Partial Product.—From the determinant expression for the third product, we know that

$$\begin{aligned} S(VAB)C &= S(VBC)A = S(VCA)B \\ &= -S(VBA)C = -S(VCB)A = -S(VAC)B. \end{aligned}$$

Hence any of the three former may be expressed by $SABC$, and any of the three latter by $-SABC$.



The third product $S(VAB)C$ is represented by the volume of the parallelepiped formed by the vectors A, B, C taken in that order. The line VAB represents in magnitude and direction the area formed by A and B , and the product of VAB with the projection of C upon it is the measure of the volume in magnitude and sign. Hence the volume formed by the three vectors has no direction in space, but it is positive or negative according to the cyclical order of the vectors.

In the expression $abc \sin \alpha \beta \cos \alpha \beta \gamma$ it is evident that $\sin \alpha \beta$ corresponds to $\sin \theta$, and $\cos \alpha \beta \gamma$ to $\cos \phi$, in the usual formula for the volume of a parallelepiped.

Example.—Let the velocity of a straight wire parallel to itself be $V = 1000/30^\circ$ centimeters per second, let the intensity of the magnetic flux be $B = 6000/90^\circ$ lines per square centimeter, and let the straight wire $L = 15$ centimeters $60^\circ/45^\circ$. Then $VVB = 6000000 \sin 60^\circ \overline{90^\circ/90^\circ}$ lines per centimeter per second. Hence $S(VVB)L = 15 \times 6000000 \sin 60^\circ \overline{\cos \phi}$ lines per second where $\cos \phi = \sin 45^\circ \sin 60^\circ$.

Sum of the Partial Vector Products.—By adding the first and second partial products we obtain the total vector product of ABC , which is denoted by $V(ABC)$. By decomposing the second product we obtain

$$V(ABC) = SAB \cdot C - SBC \cdot A + SCA \cdot B.$$

By removing the common multiplier abc , we get

$$V(\alpha\beta\gamma) = \cos \alpha\beta \cdot \gamma - \cos \beta\gamma \cdot \alpha + \cos \gamma\alpha \cdot \beta.$$

Similarly

$$V(\beta\gamma\alpha) = \cos \beta\gamma \cdot \alpha - \cos \gamma\alpha \cdot \beta + \cos \alpha\beta \cdot \gamma$$

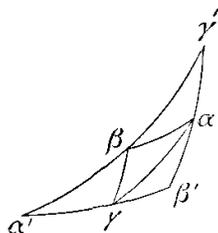
and

$$V(\gamma\alpha\beta) = \cos \gamma\alpha \cdot \beta - \cos \alpha\beta \cdot \gamma + \cos \beta\gamma \cdot \alpha.$$

These three vectors have the same magnitude, for the square of each is

$$\cos^2 \alpha\beta + \cos^2 \beta\gamma + \cos^2 \gamma\alpha - 2 \cos \alpha\beta \cos \beta\gamma \cos \gamma\alpha,$$

that is, $1 - \{S(\alpha\beta\gamma)\}^2$.



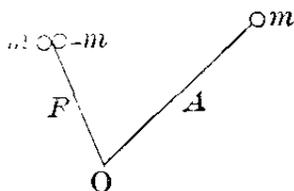
They have the directions respectively of α' , β' , γ' , which are the corners of the triangle whose sides are bisected by the corners α , β , γ of the given triangle.

- Prob. 38. Find the second partial product of $9\overline{20^\circ//30^\circ}$, $10\overline{30^\circ//40^\circ}$, $11\overline{45^\circ//45^\circ}$. Also the third partial product.
- Prob. 39. Find the cosine of the angle between the plane of $l_1i + m_1j + n_1k$ and $l_2i + m_2j + n_2k$ and the plane of $l_3i + m_3j + n_3k$ and $l_4i + m_4j + n_4k$.
- Prob. 40. Find the volume of the parallelepiped determined by the vectors $100i + 50j + 25k$, $50i + 10j + 80k$, and $-75i + 40j - 80k$.
- Prob. 41. Find the volume of the tetrahedron determined by the extremities of the following vectors: $3i - 2j + 1k$, $-4i + 5j - 7k$, $3i - 7j - 2k$, $8i + 4j - 3k$.
- Prob. 42. Find the voltage at the terminals of a conductor when its velocity is 1500 centimeters per second, the intensity of the magnetic flux is 7000 lines per square centimeter, and the length of the conductor is 20 centimeters, the angle between the first and second being 30° , and that between the plane of the first two and the direction of the third 60° . (Ans. .91 volts.)
- Prob. 43. Let $\alpha = \overline{20^\circ//10^\circ}$, $\beta = \overline{30^\circ//25^\circ}$, $\gamma = \overline{40^\circ//35^\circ}$. Find $V\alpha\beta\gamma$, and deduce $V\beta\gamma\alpha$ and $V\gamma\alpha\beta$.

Article 8

Composition of Quantities.

A number of homogeneous quantities are simultaneously located at different points; it is required to find how to add or compound them.



Addition of a Located Scalar Quantity.—Let m_A denote a mass m situated at the extremity of the radius-vector A . A mass $m - m$ may be introduced at the extremity of any radius-vector R , so that

$$\begin{aligned} m_A &= (m - m)_R + m_A \\ &= m_R + m_A - m_R \\ &= m_R + m(A - R). \end{aligned}$$

Here $A - R$ is a simultaneous sum, and denotes the radius-vector from the extremity of R to the extremity of A . The product $m(A - R)$ is what Clerk Maxwell called a mass-vector, and means the directed moment of m with respect to the extremity of R . The equation states that the mass m at the extremity of the vector A is equivalent to the equal mass at the extremity of R , together with the said mass-vector applied at the extremity of R . The equation expresses a physical of mechanical principle.

Hence for any number of masses, m_1 at the extremity of A_1 , m_2 at the extremity of A_2 , etc.,

$$\sum m_A = \sum m_R + \sum \{m(A - R)\},$$

where the latter term denotes the sum of the mass-vectors treated as simultaneous vectors applied at a common point. Since

$$\begin{aligned}\sum\{m(A-R)\} &= \sum mA - \sum mR \\ &= \sum mA - R \sum m,\end{aligned}$$

the resultant moment will vanish if

$$R = \frac{\sum mA}{\sum m}, \quad \text{or} \quad R \sum m = \sum mA$$

Corollary.—Let

$$R = xi + yj + zk,$$

and

$$A = a_1j + b_1j + c_1k;$$

then the above condition may be written as

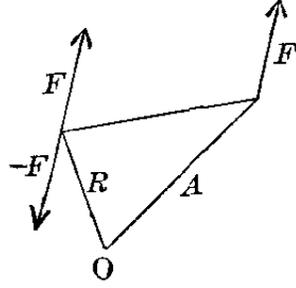
$$\begin{aligned}xi + yj + zk &= \frac{\sum\{m(ai + bj + ck)\}}{\sum m} \\ &= \frac{\sum(ma) \cdot i}{\sum m} + \frac{\sum(mb) \cdot j}{\sum m} + \frac{\sum(mc) \cdot k}{\sum m};\end{aligned}$$

therefore

$$x = \frac{\sum(ma)}{\sum m}, \quad y = \frac{\sum(mb)}{\sum m}, \quad z = \frac{\sum(mc)}{\sum m}.$$

Example.—Given 5 pounds at 10 feet $\overline{45^\circ//30^\circ}$ and 8 pounds at 7 feet $\overline{60^\circ//45^\circ}$; find the moment when both masses are transferred to 12 feet $\overline{75^\circ//60^\circ}$.

$$\begin{aligned}m_1A_1 &= 50(\cos 30^\circ i + \sin 30^\circ \cos 45^\circ j + \sin 30^\circ \sin 45^\circ k), \\ m_2A_2 &= 56(\cos 45^\circ i + \sin 45^\circ \cos 60^\circ j + \sin 45^\circ \sin 60^\circ k), \\ (m_1 + m_2)R &= 156(\cos 60^\circ i + \sin 60^\circ \cos 75^\circ j + \sin 60^\circ \sin 75^\circ k), \\ \text{moment} &= m_1A_1 + m_2A_2 - (m_1 + m_2)R.\end{aligned}$$



Composition of a Located Vector Quantity.—Let F_A denote a force applied at the extremity of the radius-vector A . As a force $F - F$ may be introduced at the extremity of any radius-vector R , we have

$$\begin{aligned} F_A &= (F - F) + F_A \\ &= F_R + \mathbf{V}(A - R)F. \end{aligned}$$

This equation asserts that a force F applied at the extremity of A is equivalent to an equal force applied at the extremity of R together with a couple whose magnitude and direction are given by the vector product of the radius-vector from the extremity of R to the extremity of A and the force.

Hence for a system of forces applied at different points, such as F_1 at A_1 , F_2 at A_2 , etc., we obtain

$$\begin{aligned} \sum (F_A) &= \sum (F_R) + \sum \mathbf{V}(A - R)F \\ &= \left(\sum F \right)_R + \sum \mathbf{V}(A - R)F. \end{aligned}$$

Since

$$\begin{aligned} \sum \mathbf{V}(A - R)F &= \sum \mathbf{V}AF - \sum \mathbf{V}RF \\ &= \sum \mathbf{V}AF - \mathbf{V}R \sum F \end{aligned}$$

the condition for no resultant couple is

$$\mathbf{V}R \sum F = \sum \mathbf{V}AF,$$

which requires $\sum F$ to be normal to $\sum \mathbf{V}AF$.

Example.—Given a force $1i + 2j + 3k$ pounds weight at $4i + 5j + 6k$ feet, and a force of $7i + 9j + 11k$ pounds weight at $10i + 12j + 14k$ feet; find the torque which must be supplied when both are transferred to $2i + 5j + 3k$, so that the

effect may be the same as before.

$$\begin{aligned} VA_1F_1 &= 3i - 6j + 3k, \\ VA_2F_2 &= 6i - 12j + 6k, \\ \sum VAF &= 9i - 18j + 9k, \\ \sum F &= 8i + 11j + 14k, \\ VR \sum F &= 37i - 4j - 18k, \\ \text{Torque} &= -28i - 14j + 27k. \end{aligned}$$

By taking the vector product of the above equal vectors with the reciprocal of $\sum F$ we obtain

$$V \left\{ (VR \sum F) \frac{1}{\sum F} \right\} = V \left\{ (\sum VAF) \frac{1}{\sum F} \right\}.$$

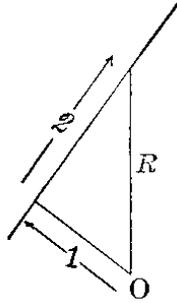
By the principle previously established the left member resolves into $-R + SR \frac{1}{\sum F} \cdot \sum F$; and the right member is equivalent to the complete product on account of the two factors being normal to one another; hence

$$-R + SR \frac{1}{\sum F} \cdot \sum F = \sum (VAF) \frac{1}{\sum F};$$

that is,

$$R = \frac{1}{\sum F} \sum (VAF) \tag{1}$$

$$+ SR \frac{1}{\sum F} \cdot \sum F. \tag{2}$$



The extremity of R lies on a straight line whose perpendicular is the vector (1) and whose direction is that of the resultant force. The term (2) means the projection of R upon that line.

The condition for the central axis is that the resultant force and the resultant couple should have the same direction; hence it is given by

$$V \left\{ \sum VAF - VR \sum F \right\} \sum F = 0;$$

that is

$$V \left(VR \sum F \right) \sum F = V \left(\sum AF \right) \sum F.$$

By expanding the left member according to the same principle as above, we obtain

$$- \left(\sum F \right)^2 R + SR \sum F \cdot \sum F = V \left(\sum AF \right) \sum F;$$

therefore

$$\begin{aligned} R &= \frac{1}{\left(\sum F \right)^2} V \sum F \left(V \sum AF \right) + \frac{SR \sum F}{\left(\sum F \right)^2} \cdot \sum F \\ &= V \left(\frac{1}{\sum F} \right) \left(V \sum AF \right) + SR \frac{1}{\sum F} \cdot \sum F. \end{aligned}$$

This is the same straight line as before, only no relation is now imposed on the directions of $\sum F$ and $\sum VAF$; hence there always is a central axis.

Example.—Find the central axis for the system of forces in the previous example. Since $\sum F = 8i + 11j + 14k$, the direction of the line is

$$\frac{8i + 11j + 14k}{\sqrt{64 + 121 + 196}}.$$

Since $\frac{1}{\sum F} = \frac{8i + 11j + 14k}{381}$ and $\sum VAF = 9i - 18j + 9k$, the perpendicular to the line is

$$V \frac{8i + 11j + 14k}{381} 9i - 18j + 9k = \frac{1}{381} \{351i + 54j - 243k\}.$$

Prob. 44. Find the moment at $\overline{90^\circ//270^\circ}$ of 10 pounds at 4 feet $\overline{10^\circ//20^\circ}$ and 20 pounds at 5 feet $\overline{30^\circ//120^\circ}$.

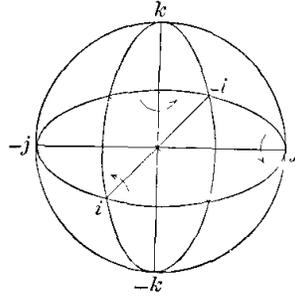
Prob. 45. Find the torque for $4i + 3j + 2k$ pounds weight at $2i - 3j + 1k$ feet, and $2i - 1k - 1k$ pounds weight at $-3i + 4j + 5k$ feet when transferred to $-3i - 2j - 4k$ feet.

Prob. 46. Find the central axis in the above case.

Prob. 47. Prove that the mass-vector drawn from any origin to a mass equal to that of the whole system placed at the center of mass of the system is equal to the sum of the mass-vectors drawn from the same origin to all the particles of the system.

Article 9

Spherical Trigonometry.



Let i, j, k denote three mutually perpendicular axes. In order to distinguish clearly between an axis and a quadrantal version round it, let $i^{\frac{\pi}{2}}, j^{\frac{\pi}{2}}, k^{\frac{\pi}{2}}$ denote quadrantal versions in the positive sense about the axes i, j, k respectively. The directions of positive version are indicated by the arrows.

By $i^{\frac{\pi}{2}}i^{\frac{\pi}{2}}$ is meant the product of two quadrantal versions round i ; it is equivalent to a semicircular version round i ; hence $i^{\frac{\pi}{2}}i^{\frac{\pi}{2}} = i^{\pi} = -$. Similarly $j^{\frac{\pi}{2}}j^{\frac{\pi}{2}}$ means the product of two quadrantal versions round j , and $j^{\frac{\pi}{2}}j^{\frac{\pi}{2}} = j^{\pi} = -$. Similarly $k^{\frac{\pi}{2}}k^{\frac{\pi}{2}} = k^{\pi} = -$.

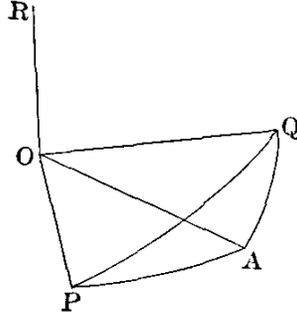
By $i^{\frac{\pi}{2}}j^{\frac{\pi}{2}}$ is meant a quadrant round i followed by a quadrant round j ; it is equivalent to the quadrant from j to i , that is, to $-k^{\frac{\pi}{2}}$. But $j^{\frac{\pi}{2}}i^{\frac{\pi}{2}}$ is equivalent to the quadrant from $-i$ to $-j$, that is, to $k^{\frac{\pi}{2}}$. Similarly for the other two pairs of products. Hence we obtain the following

Rules for Versors.

$$\begin{aligned}
 i^{\frac{\pi}{2}}i^{\frac{\pi}{2}} &= -, & j^{\frac{\pi}{2}}j^{\frac{\pi}{2}} &= -, & k^{\frac{\pi}{2}}k^{\frac{\pi}{2}} &= -, \\
 i^{\frac{\pi}{2}}j^{\frac{\pi}{2}} &= -k^{\frac{\pi}{2}}, & j^{\frac{\pi}{2}}i^{\frac{\pi}{2}} &= k^{\frac{\pi}{2}}, \\
 j^{\frac{\pi}{2}}k^{\frac{\pi}{2}} &= -i^{\frac{\pi}{2}}, & k^{\frac{\pi}{2}}j^{\frac{\pi}{2}} &= i^{\frac{\pi}{2}}, \\
 k^{\frac{\pi}{2}}i^{\frac{\pi}{2}} &= -j^{\frac{\pi}{2}}, & i^{\frac{\pi}{2}}k^{\frac{\pi}{2}} &= j^{\frac{\pi}{2}}.
 \end{aligned}$$

The meaning of these rules will be seen from the following application. Let $li + mj + nk$ denote any axis, then $(li + mj + nk)^{\frac{\pi}{2}}$ denotes a quadrant of angle round that axis. This quadrantal version can be decomposed into the three rectangular components $li^{\frac{\pi}{2}}$, $mj^{\frac{\pi}{2}}$, $nk^{\frac{\pi}{2}}$; and these components are not successive versions, but the parts of one version. Similarly any other quadrantal version $(l'i + m'j + n'k)^{\frac{\pi}{2}}$ can be resolved into $l'i^{\frac{\pi}{2}}$, $m'j^{\frac{\pi}{2}}$, $n'k^{\frac{\pi}{2}}$. By applying the above rules, we obtain

$$\begin{aligned} & (li + mj + nk)^{\frac{\pi}{2}}(l'i + m'j + n'k)^{\frac{\pi}{2}} \\ &= (li^{\frac{\pi}{2}} + mj^{\frac{\pi}{2}} + nk^{\frac{\pi}{2}})(l'i^{\frac{\pi}{2}} + m'j^{\frac{\pi}{2}} + n'k^{\frac{\pi}{2}}) \\ &= -(ll' + mm' + nn') - (mn' - m'n)i^{\frac{\pi}{2}} - (nl' - n'l)j^{\frac{\pi}{2}} - (lm' - l'm)k^{\frac{\pi}{2}} \\ &= -(ll' + mm' + nn') - \{(mn' - m'n)i + (nl' - n'l)j + (lm' - l'm)k\}^{\frac{\pi}{2}}. \end{aligned}$$



Product of Two Spherical Versors.—Let β denote the axis and b the ratio of the spherical versor PA , then the versor itself is expressed by β^b . Similarly let γ denote the axis and c the ratio of the spherical versor AQ , then the versor itself is expressed by γ^c .

Now

$$\beta^b = \cos b + \sin b \cdot \beta^{\frac{\pi}{2}},$$

and

$$\gamma^c = \cos c + \sin c \cdot \gamma^{\frac{\pi}{2}};$$

therefore

$$\begin{aligned} \beta^b \gamma^c &= (\cos b + \sin b \cdot \beta^{\frac{\pi}{2}})(\cos c + \sin c \cdot \gamma^{\frac{\pi}{2}}) \\ &= \cos b \cos c + \cos b \sin c \cdot \gamma^{\frac{\pi}{2}} + \cos c \sin b \cdot \beta^{\frac{\pi}{2}} + \sin b \sin c \cdot \beta^{\frac{\pi}{2}} \gamma^{\frac{\pi}{2}}. \end{aligned}$$

But from the preceding paragraph

$$\beta^{\frac{\pi}{2}} \gamma^{\frac{\pi}{2}} = -\cos \beta\gamma - \sin \beta\gamma \cdot \overline{\beta\gamma}^{\frac{\pi}{2}};$$

therefore

$$\beta^b \gamma^c = \cos b \cos c - \sin b \sin c \cos \beta\gamma \quad (1)$$

$$+ \{ \cos b \sin c \cdot \gamma + \cos c \sin b \cdot \beta - \sin b \sin c \sin \beta\gamma \cdot \overline{\beta\gamma} \}^{\frac{\pi}{2}}. \quad (2)$$

The first term gives the cosine of the product versor; it is equivalent to the fundamental theorem of spherical trigonometry, namely,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

where A denotes the external angle instead of the angle included by the sides.

The second term is the directed sine of the angle; for the square of (2) is equal to 1 minus the square of (1), and its direction is normal to the plane of the product angle.¹

Example.—Let $\beta = \overline{30^\circ} // \underline{45^\circ}$ and $\gamma = \overline{60^\circ} // \underline{30^\circ}$. Then

$$\cos \beta\gamma = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \cos 30^\circ,$$

and

$$\sin \beta\gamma \cdot \overline{\beta\gamma} = V\beta\gamma;$$

but

$$\beta = \cos 45^\circ i + \sin 45^\circ \cos 30^\circ j + \sin 45^\circ \sin 30^\circ k,$$

and

$$\gamma = \cos 30^\circ i + \sin 30^\circ \cos 60^\circ j + \sin 30^\circ \sin 60^\circ k;$$

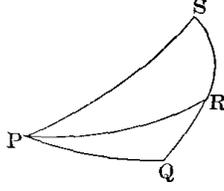
therefore

$$\begin{aligned} V\beta\gamma &= \{ \sin 45^\circ \cos 30^\circ \sin 30^\circ \sin 60^\circ - \sin 45^\circ \sin 30^\circ \sin 30^\circ \cos 60^\circ \} i \\ &\quad + \{ \sin 45^\circ \sin 30^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \sin 60^\circ \} j \\ &\quad + \{ \cos 45^\circ \sin 30^\circ \cos 60^\circ - \sin 45^\circ \cos 30^\circ \cos 30^\circ \} k. \end{aligned}$$

Quotient of Two Spherical Versors.—The reciprocal of a given versor is derived by changing the sign of the index; γ^{-c} is the reciprocal of γ^c . As $\beta^b = \cos b + \sin b \cdot \beta^{\frac{\pi}{2}}$, and $\gamma^{-c} = \cos c - \sin c \cdot \gamma^{\frac{\pi}{2}}$,

$$\begin{aligned} \beta^b \gamma^{-c} &= \cos b \cos c + \sin b \sin c \cos \beta\gamma \\ &\quad + \{ \cos c \sin b \cdot \beta - \cos b \sin c \cdot \gamma + \sin b \sin c \sin \beta\gamma \cdot \overline{\beta\gamma} \}^{\frac{\pi}{2}}. \end{aligned}$$

¹Principles of Elliptic and Hyperbolic Analysis, p. 2.



Product of Three Spherical Versors.—Let α^a denote the versor PQ , β^b the versor QR , and γ^c the versor RS ; then $\alpha^a\beta^b\gamma^c$ denotes PS . Now $\alpha^a\beta^b\gamma^c$

$$= (\cos a + \sin a \cdot \alpha^{\frac{\pi}{2}})(\cos b + \sin b \cdot \beta^{\frac{\pi}{2}})(\cos c + \sin c \cdot \gamma^{\frac{\pi}{2}})$$

$$= \cos a \cos b \cos c \tag{1}$$

$$+ \cos a \cos b \sin c \cdot \gamma^{\frac{\pi}{2}} + \cos a \cos c \sin b \cdot \beta^{\frac{\pi}{2}} + \cos b \cos c \sin a \cdot \alpha^{\frac{\pi}{2}} \tag{2}$$

$$+ \cos a \sin b \sin c \cdot \beta^{\frac{\pi}{2}} \gamma^{\frac{\pi}{2}} + \cos b \sin a \sin c \cdot \alpha^{\frac{\pi}{2}} \gamma^{\frac{\pi}{2}}$$

$$+ \cos c \sin a \sin b \cdot \alpha^{\frac{\pi}{2}} \beta^{\frac{\pi}{2}} \tag{3}$$

$$+ \sin a \sin b \sin c \cdot \alpha^{\frac{\pi}{2}} \beta^{\frac{\pi}{2}} \gamma^{\frac{\pi}{2}} \tag{4}$$

The versors in (3) are expanded by the rule already obtained, namely,

$$\beta^{\frac{\pi}{2}} \gamma^{\frac{\pi}{2}} = -\cos \beta\gamma - \sin \beta\gamma \cdot \overline{\beta\gamma^{\frac{\pi}{2}}}.$$

The versor of the fourth term is

$$\alpha^{\frac{\pi}{2}} \beta^{\frac{\pi}{2}} \gamma^{\frac{\pi}{2}} = -(\cos \alpha\beta + \sin \alpha\beta \cdot \overline{\alpha\beta^{\frac{\pi}{2}}}) \gamma^{\frac{\pi}{2}}$$

$$= -\cos \alpha\beta \cdot \gamma^{\frac{\pi}{2}} + \sin \alpha\beta \cos \overline{\alpha\beta\gamma} + \sin \alpha\beta \sin \overline{\alpha\beta\gamma} \cdot \overline{\overline{\alpha\beta\gamma}}^{\frac{\pi}{2}}.$$

Now $\sin \alpha\beta \sin \overline{\alpha\beta\gamma} \cdot \overline{\overline{\alpha\beta\gamma}} = \cos \alpha\gamma \cdot \beta - \cos \beta\gamma \cdot \alpha$ (p. 451), hence the last term of the product, when expanded, is

$$\sin a \sin b \sin c \{ -\cos \alpha\beta \cdot \gamma^{\frac{\pi}{2}} + \cos \alpha\gamma \cdot \beta^{\frac{\pi}{2}} - \cos \beta\gamma \cdot \alpha^{\frac{\pi}{2}} + \cos \overline{\alpha\beta\gamma} \}.$$

Hence

$$\begin{aligned}\cos \alpha^a \beta^b \gamma^c &= \cos a \cos b \cos c - \cos a \sin b \sin c \cos \beta\gamma \\ &\quad - \cos b \sin a \sin c \cos \alpha\gamma - \cos c \sin a \sin b \cos \alpha\beta \\ &\quad + \sin a \sin b \sin c \sin \alpha\beta \cos \alpha\beta\gamma,\end{aligned}$$

and, letting Sin denote the directed sine,

$$\begin{aligned}\text{Sin } \alpha^a \beta^b \gamma^c &= \cos a \cos b \sin c \cdot \gamma + \cos a \cos c \sin b \cdot \beta \\ &\quad + \cos b \cos c \sin a \cdot \alpha - \cos a \sin b \sin c \sin \beta\gamma \cdot \overline{\beta\gamma} \\ &\quad - \cos b \sin a \sin c \sin \alpha\gamma \cdot \overline{\alpha\gamma} \\ &\quad - \cos c \sin a \sin b \sin \alpha\beta \cdot \overline{\alpha\beta} \\ &\quad - \sin a \sin b \sin c \{ \cos \alpha\beta \cdot \gamma - \cos \alpha\gamma \cdot \beta + \cos \beta\gamma \cdot \alpha \} .^2\end{aligned}$$

Extension of the Exponential Theorem to Spherical Trigonometry.—It has been shown (p. 458) that

$$\cos \beta^b \gamma^c = \cos b \cos c - \sin b \sin c \cos \beta\gamma$$

and

$$(\sin \beta^b \gamma^c)^{\frac{\pi}{2}} = \cos c \sin b \cdot \beta^{\frac{\pi}{2}} + \cos b \sin c \cdot \gamma^{\frac{\pi}{2}} - \sin b \sin c \sin \beta\gamma \cdot \overline{\beta\gamma}^{\frac{\pi}{2}}.$$

Now

$$\cos b = 1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \frac{b^6}{6!} + \text{etc.}$$

and

$$\sin b = b - \frac{b^3}{3!} + \frac{b^5}{5!} - \text{etc.}$$

Substitute these series for $\cos b$, $\sin b$, $\cos c$, and $\sin c$ in the above equations, multiply out, and group the homogeneous terms together. It will be found that

$$\begin{aligned}\cos \beta^b \gamma^c &= 1 - \frac{1}{2!} \{ b^2 + 2bc \cos \beta\gamma + c^2 \} \\ &\quad + \frac{1}{4!} \{ b^4 + 4b^3 c \cos \beta\gamma + 6b^2 c^2 + 4bc^3 \cos \beta\gamma + c^4 \} \\ &\quad - \frac{1}{6!} \{ b^6 + 6b^5 c \cos \beta\gamma + 15b^4 c^2 + 20b^3 c^3 \cos \beta\gamma \\ &\quad \quad + 15b^2 c^4 + 6bc^5 \cos \beta\gamma + c^6 \} + \dots,\end{aligned}$$

²In the above case the three axes of the successive angles are not perfectly independent, for the third angle must begin where the second leaves off. But the theorem remains true when the axes are independent; the factors are then quaternions in the most general sense.

where the coefficients are those of the binomial theorem, the only difference being that $\cos \beta\gamma$ occurs in all the odd terms as a factor. Similarly, by expanding the terms of the sine, we obtain

$$\begin{aligned} (\sin \beta^b \gamma^c)^{\frac{\pi}{2}} &= b \cdot \beta^{\frac{\pi}{2}} + c \cdot \gamma^{\frac{\pi}{2}} - bc \sin \beta\gamma \cdot \overline{\beta\gamma^{\frac{\pi}{2}}} \\ &\quad - \frac{1}{3!} \{b^3 \cdot \beta^{\frac{\pi}{2}} + 3b^2 c \cdot \gamma^{\frac{\pi}{2}} + 3bc^2 \cdot \beta^{\frac{\pi}{2}} + c^3 \cdot \gamma^{\frac{\pi}{2}}\} \\ &\quad + \frac{1}{3!} \{bc^3 + b^3 c\} \sin \beta\gamma \cdot \overline{\beta\gamma^{\frac{\pi}{2}}} \\ &\quad + \frac{1}{5!} \{b^5 \cdot \beta^{\frac{\pi}{2}} + 5b^4 c \cdot \gamma^{\frac{\pi}{2}} + 10b^3 c^2 \cdot \beta^{\frac{\pi}{2}} \\ &\quad \quad + 10b^2 c^3 \cdot \gamma^{\frac{\pi}{2}} + 5bc^4 \cdot \beta^{\frac{\pi}{2}} + c^5 \cdot \gamma^{\frac{\pi}{2}} \\ &\quad - \frac{1}{5!} \left\{ b^5 c + \frac{5 \cdot 4}{2 \cdot 3} b^2 c^3 + bc^5 \right\} \sin \beta\gamma \cdot \overline{\beta\gamma^{\frac{\pi}{2}}} - \dots \end{aligned}$$

By adding these two expansions together we get the expansion for $\beta^b \gamma^c$, namely,

$$\begin{aligned} \beta^b \gamma^c &= 1 + b \cdot \beta^{\frac{\pi}{2}} + c \cdot \gamma^{\frac{\pi}{2}} \\ &\quad - \frac{1}{2!} \{b^2 + 2bc(\cos \beta\gamma + \sin \beta\gamma \cdot \overline{\beta\gamma^{\frac{\pi}{2}}}) + c^2\} \\ &\quad - \frac{1}{3!} \{b^3 \cdot \beta^{\frac{\pi}{2}} + 3b^2 c \cdot \gamma^{\frac{\pi}{2}} + 3bc^2 \cdot \beta^{\frac{\pi}{2}} + c^3 \cdot \gamma^{\frac{\pi}{2}}\} \\ &\quad + \frac{1}{4!} \{b^4 + 4b^3 c(\cos \beta\gamma + \sin \beta\gamma \cdot \overline{\beta\gamma^{\frac{\pi}{2}}}) + 6b^2 c^2 \\ &\quad \quad + 4bc^3(\cos \beta\gamma + \sin \beta\gamma \cdot \overline{\beta\gamma^{\frac{\pi}{2}}}) + c^4\} + \dots \end{aligned}$$

By restoring the minus, we find that the terms on the second line can be thrown into the form

$$\frac{1}{2!} \{b^2 \cdot \beta^{\pi} + 2bc \cdot \beta^{\frac{\pi}{2}} \gamma^{\frac{\pi}{2}} + c^2 \cdot \gamma^{\pi}\},$$

and this is equal to

$$\frac{1}{2!} \{b \cdot \beta^{\frac{\pi}{2}} + c \cdot \gamma^{\frac{\pi}{2}}\}^2,$$

where we have the square of a sum of successive terms. In a similar manner the terms on the third line can be restored to

$$b^3 \cdot \beta^{\frac{3\pi}{2}} + 3b^2 c \cdot \beta^{\pi} \gamma^{\frac{\pi}{2}} + 3bc^2 \cdot \beta^{\frac{\pi}{2}} \gamma^{\pi} + c^3 \cdot \gamma^{3(\frac{\pi}{2})},$$

that is,

$$\frac{1}{3!} \{b \cdot \beta^{\frac{\pi}{2}} + c \cdot \gamma^{\frac{\pi}{2}}\}^3.$$

Hence

$$\begin{aligned}\beta^b \gamma^c &= 1 + b \cdot \beta^{\frac{\pi}{2}} + c \cdot \gamma^{\frac{\pi}{2}} + \frac{1}{2!} \{b \cdot \beta^{\frac{\pi}{2}} + c \cdot \gamma^{\frac{\pi}{2}}\}^2 \\ &\quad + \frac{1}{3!} \{b \cdot \beta^{\frac{\pi}{2}} + c \cdot \gamma^{\frac{\pi}{2}}\}^3 + \frac{1}{4!} \{b \cdot \beta^{\frac{\pi}{2}} + c \cdot \gamma^{\frac{\pi}{2}}\}^4 + \dots \\ &= e^{b \cdot \beta^{\frac{\pi}{2}} + c \cdot \gamma^{\frac{\pi}{2}}}. \quad 3\end{aligned}$$

Extension of the Binomial Theorem.—We have proved above that $e^{b\beta^{\frac{\pi}{2}}} e^{c\gamma^{\frac{\pi}{2}}} = e^{b\beta^{\frac{\pi}{2}} + c\gamma^{\frac{\pi}{2}}}$ provided that the powers of the binomial are expanded as due to a successive sum, that is, the order of the terms in the binomial must be preserved. Hence the expansion for a power of a successive binomial is given by

$$\begin{aligned}\{b \cdot \beta^{\frac{\pi}{2}} + c \cdot \gamma^{\frac{\pi}{2}}\}^n &= b^n \cdot \beta^{n \cdot \frac{\pi}{2}} + n b^{n-1} c \cdot \beta^{(n-1) \cdot (\frac{\pi}{2})} \gamma^{\frac{\pi}{2}} \\ &\quad + \frac{n(n-1)}{1 \cdot 2} b^{n-2} c^2 \cdot \beta^{(n-2) \cdot (\frac{\pi}{2})} \gamma^{\pi} + \text{etc.}\end{aligned}$$

Example.—Let $b = \frac{1}{10}$ and $c = \frac{1}{5}$, $\beta = \overline{30^\circ // 45^\circ}$, $\gamma = \overline{60^\circ // 30^\circ}$.

$$\begin{aligned}(b \cdot \beta^{\frac{\pi}{2}} + c \cdot \gamma^{\frac{\pi}{2}})^2 &= -\{b^2 + c^2 + 2bc \cos \beta\gamma + 2bc(\sin \beta\gamma)^{\frac{\pi}{2}}\} \\ &= -\left(\frac{1}{100} + \frac{1}{25} + \frac{2}{50} \cos \beta\gamma\right) - \frac{2}{50} (\sin \beta\gamma)^{\frac{\pi}{2}}.\end{aligned}$$

Substitute the calculated values of $\cos \beta\gamma$ and $\sin \beta\gamma$ (page 459).

Prob. 48. Find the equivalent of a quadrantal version round $\frac{\sqrt{3}}{2}i + \frac{1}{2\sqrt{2}}j + \frac{1}{2\sqrt{2}}k$ followed

by a quadrantal version round $\frac{1}{2}i + \frac{\sqrt{3}}{4}j + \frac{3}{4}k$.

Prob. 49. In the example on p. 459 let $b = 25^\circ$ and $c = 50^\circ$; calculate out the cosine and the directed sine of the product angle.

Prob. 50. In the above example calculate the cosine and the directed sine up to and inclusive of the fourth power of the binomial. (Ans. $\cos = .9735$.)

Prob. 51. Calculate the first four terms of the series when $b = \frac{1}{50}$, $c = \frac{1}{100}$, $\beta = \overline{0^\circ // 0^\circ}$, $\gamma = \overline{90^\circ // 90^\circ}$.

Prob. 52. From the fundamental theorem of spherical trigonometry deduce the polar theorem with respect to both the cosine and the directed sine.

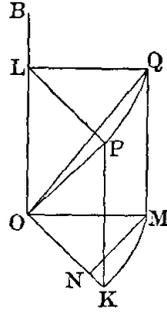
Prob. 53. Prove that if $\alpha^a, \beta^b, \gamma^c$ denote the three versors of a spherical triangle, then

$$\frac{\sin \beta\gamma}{\sin a} = \frac{\sin \gamma\alpha}{\sin b} = \frac{\sin \alpha\beta}{\sin c}.$$

³At page 386 of his Elements of Quaternions, Hamilton says: "In the present theory of diplanar quaternions we cannot expect to find that the sum of the logarithms of any two proposed factors shall be generally equal to the logarithm of the product; but for the simpler and earlier case of coplanar quaternions, that algebraic property may be considered to exist, with due modification for multiplicity of value." He was led to this view by not distinguishing between vectors and quadrantal quaternions and between simultaneous and successive addition. The above demonstration was first given in my paper on "The Fundamental Theorems of Analysis generalized for Space." It forms the key to the higher development of space analysis.

Article 10

Composition of Rotations.



A version refers to the change of direction of a line, but a rotation refers to a rigid body. The composition of rotations is a different matter from the composition of versions.

Effect of a Finite Rotation on a Line.—Suppose that a rigid body rotates θ radians round the axis β passing through the point O , and that R is the radius vector from O to some particle. In the diagram OB represents the axis β , and OP the vector R . Draw OK and OL , the rectangular components of R .

$$\begin{aligned}\beta^\theta R &= (\cos \theta + \sin \theta \cdot \beta^{\frac{\pi}{2}})r\rho \\ &= r(\cos \theta + \sin \theta \cdot \beta^{\frac{\pi}{2}})(\cos \beta\rho \cdot \beta + \sin \beta\rho \cdot \overline{\beta\rho\beta}) \\ &= r\{\cos \beta\rho \cdot \beta + \cos \theta \sin \beta\rho \cdot \overline{\beta\rho\beta} + \sin \theta \sin \beta\rho \cdot \overline{\beta\rho}\}.\end{aligned}$$

When $\cos \beta\rho = 0$, this reduces to

$$\beta^\theta R = \cos \theta R + \sin \theta V(\beta R).$$

The general result may be written

$$\beta^\theta R = S\beta R \cdot \beta + \cos \theta (V\beta R)\beta + \sin \theta V\beta R.$$

Note that $(V\beta R)\beta$ is equal to $V(V\beta R)\beta$ because $S\beta R\beta$ is 0, for it involves two coincident directions.

Example.—Let $\beta = li + mj + nk$, where $l^2 + m^2 + n^2 = 1$ and $R = xi + yj + zk$; then $S\beta R = lx + my + nz$

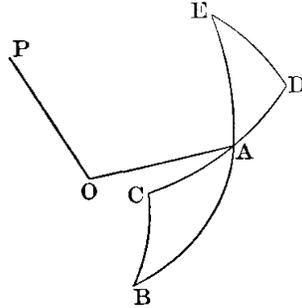
$$V(\beta R)\beta = \begin{vmatrix} mz - ny & nx - lz & ly - mx \\ l & m & n \\ i & j & k \end{vmatrix}$$

and

$$V\beta R = \begin{vmatrix} l & m & n \\ x & y & z \\ i & j & k \end{vmatrix}.$$

Hence

$$\begin{aligned} \beta^\theta &= (lx + my + nz)(li + mj + nk) \\ &+ \cos \theta \begin{vmatrix} mz - ny & nx - lz & ly - mx \\ l & m & n \\ i & j & k \end{vmatrix} \\ &+ \sin \theta \begin{vmatrix} l & m & n \\ x & y & z \\ i & j & k \end{vmatrix}. \end{aligned}$$



To prove that $\beta^b \rho$ coincides with the axis of $\beta^{-\frac{b}{2}} \rho^{\frac{\pi}{2}} \beta^{\frac{b}{2}}$. Take the more general versor ρ^θ . Let OP represent the axis β , AB the versor $\beta^{-\frac{b}{2}}$, BC the versor ρ^θ . Then $(AB)(BC) = AC = DA$, therefore $(AB)(BC)(AE) = (DA)(AE) = DE$. Now DE has the same angle as BC , but its axis has been rotated round P by the angle b . Hence if $\theta = \frac{\pi}{2}$, the axis of $\beta^{-\frac{b}{2}} \rho^{\frac{\pi}{2}} \beta^{\frac{b}{2}}$ will coincide with $\beta^b \rho$.¹

The exponential expression for $\beta^{-\frac{b}{2}} \rho^{\frac{\pi}{2}} \beta^{\frac{b}{2}}$ is $e^{-\frac{1}{2}b\beta^{\frac{\pi}{2}} + \frac{1}{2}\pi\rho^{\frac{\pi}{2}} + \frac{1}{2}b\beta^{\frac{\pi}{2}}}$ which may be expanded according to the exponential theorem, the successive powers of the

¹This theorem was discovered by Cayley. It indicates that quaternion multiplication in the most general sense has its physical meaning in the composition of rotations.

trinomial being formed according to the multinomial theorem, the order of the factors being preserved.

Composition of Finite Rotations round Axes which Intersect.—Let β and γ denote the two axes in space round which the successive rotations take place, and let β^b denote the first and γ^c the second. Let $\beta^b \times \gamma^c$ denote the single rotation which is equivalent to the two given rotations applied in succession; the sign \times is introduced to distinguish from the product of versors. It has been shown in the preceding paragraph that

$$\beta^b \rho = \beta^{-\frac{b}{2}} \rho^{\frac{\pi}{2}} \beta^{\frac{b}{2}};$$

and as the result is a line, the same principle applies to the subsequent rotation. Hence

$$\begin{aligned} \gamma^c(\beta^b \rho) &= \gamma^{-\frac{c}{2}} (\beta^{-\frac{b}{2}} \rho^{\frac{\pi}{2}} \beta^{\frac{b}{2}}) \gamma^{\frac{c}{2}} \\ &= (\gamma^{-\frac{c}{2}} \beta^{-\frac{b}{2}}) \rho^{\frac{\pi}{2}} (\beta^{\frac{b}{2}} \gamma^{\frac{c}{2}}), \end{aligned}$$

because the factors in a product of versors can be associated in any manner. Hence, reasoning backwards,

$$\beta^b \times \gamma^c = (\beta^{\frac{b}{2}} \gamma^{\frac{c}{2}})^2.$$

Let m denote the cosine of $\beta^{\frac{b}{2}} \gamma^{\frac{c}{2}}$, namely,

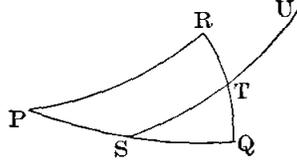
$$\cos \frac{b}{2} \cos \frac{c}{2} - \sin \frac{b}{2} \sin \frac{c}{2},$$

and $n \cdot \nu$ their directed sine, namely,

$$\cos \frac{b}{2} \sin \frac{c}{2} \cdot \gamma + \cos \frac{c}{2} \sin \frac{b}{2} \cdot \beta - \sin \frac{b}{2} \sin \frac{c}{2} \sin \beta \gamma \cdot \overline{\beta \gamma};$$

then

$$\beta^b \times \gamma^c = m^2 - n^2 + 2mn \cdot \nu.$$



Observation.—The expression $(\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}})^2$ is not, as might be supposed, identical with $\beta^b\gamma^c$. The former reduces to the latter only when β and γ are the same or opposite. In the figure β^b is represented by PQ , γ^c by QR , $\beta^b\gamma^c$ by PR , $\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}$ by ST , and $(\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}})^2$ by SU , which is twice ST . The cosine of SU differs from the cosine of PR by the term $-(\sin \frac{b}{2} \sin \frac{c}{2} \sin \beta\gamma)^2$. It is evident from the figure that their axes are also different.

Corollary.—When b and c are infinitesimals, $\cos \beta^b \times \gamma^c = 1$, and $\text{Sin } \beta^b \times \gamma^c = b \cdot \beta + c \cdot \gamma$, which is the parallelogram rule for the composition of infinitesimal rotations.

- Prob. 54. Let $\beta = \overline{30^\circ//45^\circ}$, $\theta = \frac{\pi}{3}$, and $R = 2i - 3j + 4k$; calculate $\beta^\theta R$.
- Prob. 55. Let $\beta = \overline{90^\circ//90^\circ}$, $\theta = \frac{\pi}{4}$, $R = -i + 2j - 3k$; calculate $\beta^\theta R$.
- Prob. 56. Prove by multiplying out that $\beta^{\frac{-b}{2}} \rho^{\frac{\pi}{2}} \beta^{\frac{b}{2}} = \{\beta^b \rho\}^{\frac{\pi}{2}}$.
- Prob. 57. Prove by means of the exponential theorem that $\gamma^{-c} \beta^b \gamma^c$ has an angle b , and that its axis is $\gamma^{2c} \beta$.
- Prob. 58. Prove that the cosine of $(\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}})^2$ differs from the cosine of $\beta^b\gamma^c$ by $-(\sin \frac{b}{2} \sin \frac{c}{2} \sin \beta\gamma)^2$.
- Prob. 59. Compare the axes of $(\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}})^2$ and $\beta^b\gamma^c$.
- Prob. 60. Find the value of $\beta^b \times \gamma^c$ when $\beta = \overline{0^\circ//90^\circ}$ and $\gamma = \overline{90^\circ//90^\circ}$.
- Prob. 61. Find the single rotation equivalent to $i^{\frac{\pi}{2}} \times j^{\frac{\pi}{2}} \times k^{\frac{\pi}{2}}$.
- Prob. 62. Prove that successive rotations about radii to two corners of a spherical triangle and through angles double of those of the triangle are equivalent to a single rotation about the radius to the third corner, and through an angle double of the external angle of the triangle.

Index

- Algebra
 - of space, 1
 - of the plane, 1
- Algebraic imaginary, 17
- Argand method, 19
- Association of three vectors, 13
- Bibliography, iv, 1
- Binomial theorem in spherical analysis, 41
- Cartesian analysis, 1
- Cayley, 45
- Central axis, 35
- Coaxial Quaternions, 16
 - Addition of, 18
 - Product of, 19
 - Quotient of, 19
- Complete product
 - of three vectors, 28
 - of two vectors, 9, 24
- Components
 - of quaternion, 17
 - of reciprocal of quaternion, 17
 - of versor, 16
- Composition
 - of any number of simultaneous components, 6
 - of coaxial quaternions, 18
 - of finite rotations, 44
 - of located vectors, 34
 - of mass-vectors, 32
 - of simultaneous vectors in space, 21
 - of successive components, 7
 - of two simultaneous components, 4
- Coplanar vectors, 9
- Couple of forces, 33
 - condition for couple vanishing, 33
- Cyclical and natural order, 15
- Determinant
 - for scalar product of three vectors, 30
 - for second partial product of three vectors, 28, 30
 - for vector product of two vectors, 25
- Distributive rule, 27
- Dynamo rule, 24
- Electric motor rule, 24
- Exponential theorem in spherical trigonometry, 41
 - Hamilton's view, 43
- Hamilton's
 - analysis of vector, 3
 - idea of quaternion, 16
 - view of exponential theorem in spherical analysis, 43
- Hayward, 19
- Hospitalier system, 3
- Imaginary algebraic, 17
- Kennelly's notation, 3
- Located vectors, 34
- Mass-vector, 32
 - composition of, 32
- Maxwell, 32
- Meaning

- of \angle , 3
 - of $\frac{1}{2}\pi$ as index, 37
 - of ∇ , 21
 - of dot, 3
 - of S, 10
 - of V, 11
 - of vinculum over two axes, 11
- Natural order, 15
- Notation for vector, 3
- Opposite vector, 12
- Parallelogram of simultaneous components, 4
- Partial products, 9, 25
 - of three vectors, 29
 - resolution of second partial product, 29
- Polygon of simultaneous components, 6
- Product
 - complete, 9, 24
 - of coaxial quaternions, 19
 - of three coplanar vectors, 13
 - of three spherical versors, 40
 - of two coplanar vectors, 9
 - of two quadrantal versors, 38
 - of two spherical versors, 38
 - of two sums of simultaneous vectors, 26
 - of two vectors in space, 25
 - partial, 9, 25
 - scalar, 10, 25
 - vector, 11, 25
- Quadrantal versor, 12
- Quaternion
 - definition of, 16
 - etymology of, 16
 - reciprocal of, 17
- Quaternions
 - Coaxial, 16
 - definon of, 1
 - relation to vector analysis, 1
- Quotient of two coaxial quaternions, 19
- Rayleigh, 18
- Reciprocal
 - of a quaternion, 17
 - of a vector, 12
- Relation of right-handed screw, 24
- Resolution
 - of a vector, 5
 - of second partial product of three vectors, 29
- Rotations, finite, 44
- Rules
 - for dynamo, 24
 - for expansion of product of two quadrantal versors, 38
 - for vectors, 9, 23
 - for versors, 37
- Scalar product, 10
 - geometrical meaning, 10
 - of two coplanar vectors, 9
- Screw, relation of right-handed, 24
- Simultaneous components, 3
 - composition of, 4
 - parallelogram of, 4
 - polygon of, 6
 - product of two sums of, 26
 - resolution of, 5
- Space-analysis, 1
 - advantage over Cartesian analysis, 1
 - foundation of, 7
- Spherical trigonometry, 37
 - binomial theorem, 41
 - fundamental theorem of, 39
- Spherical versor, 38
 - product of three, 40
 - product of two, 38
 - quotient of two, 39
- Square
 - of a vector, 10
 - of three successive components, 14
 - of two simultaneous components, 14
 - of two successive components, 14
- Stringham, 19
- Successive components, 4

- composition of, 7
- Tait's analysis of vector, 3
- Torque, 35
- Total vector product of three vectors,
31
- Unit-vector, 3
- Vector
 - co-planar, 9
 - definition of, 3
 - dimensions of, 3
 - in space, 21
 - notation for, 3
 - opposite of, 12
 - reciprocal of, 12
 - simultaneous, 3
 - successive, 4
- Vector analysis
 - definition of, 1
 - relation to Quaternions, 1
- Vector product, 11
 - of three vectors, 31
 - of two vectors, 11
- Versor
 - components of, 16, 38
 - product of three general spherical,
40
 - product of two general spherical,
38
 - product of two quadrantal, 38
 - rules for, 37

SHORT-TITLE CATALOGUE
OF THE
PUBLICATIONS
OF
JOHN WILEY & SONS, Inc.
NEW YORK
London: CHAPMAN & HALL, Limited
Montreal, Can.: RENOUF PUB. CO.

ARRANGED UNDER SUBJECTS

Descriptive circulars sent on application. Books marked with an asterisk (*) are sold at *net* prices only. All books are bound in cloth unless otherwise stated.

AGRICULTURE—HORTICULTURE—FORESTRY.

ARMSBY—Principles of Animal Nutrition	8vo,	\$4 00
BOWMAN—Forest Physiography	8vo,	*5 00
BRYANT—Hand Book of Logging	(Ready, Fall 1913)	
BUDD and HANSEN—American Horticultural Manual:		
Part I. Propagation, Culture, and Improvement	12mo,	1 50
Part II. Systematic Pomology	12mo,	1 50
ELLIOTT—Engineering for Land Drainage	12mo,	2 00
Practical Farm Drainage. (Second Edition, Rewritten)	12mo,	1 50
FULLER—Domestic Water Supplies for the Farm	8vo,	*1 50
GRAHAM—Text-book on Poultry	(<i>In Preparation.</i>)	
Manual on Poultry (Loose Leaf Lab. Manual)	(<i>In Preparation.</i>)	
GRAVES—Forest Mensuration	8vo,	4 00
Principles of Handling Woodlands	Small 8vo,	*1 50
GREEN—Principles of American Forestry	12mo,	1 50
GROTFENFELT—Principles of Modern Dairy Practice. (WOLL.)	12mo,	2 00
HAWLEY and HAWES—Forestry in New England	8vo,	*3 50
HERRICK—Denatured or Industrial Alcohol	8vo,	*4 00
HOWE—Agricultural Drafting	oblong quarto,	*1 25
Reference and Problem Sheets to accompany Agricultural Drafting, . . . each		*0 20
KEITT—Agricultural Chemistry Text-book	(<i>In Preparation.</i>)	
Laboratory and Field Exercises in Agricultural Chemistry . (<i>In Preparation.</i>)		
KEMP and WAUGH—Landscape Gardening. (New Edition, Rewritten) . . .	12mo,	*1 50
LARSEN—Exercises in Dairying (Loose Leaf Field Manual)	4to, paper,	*1 00
Single Exercises each		*0 02
and WHITE—Dairy Technology	Small 8vo,	*1 50
LEVISON—Studies of Trees, Loose Leaf Field Manual, 4to, pamphlet form,		
Price from 5–10 cents net, each, according to number of pages.		
MCCALL—Crops and Soils (Loose Leaf Field Manual)	(<i>In Preparation.</i>)	
Soils (Text-book)	(<i>In Preparation.</i>)	
MCKAY and LARSEN—Principles and Practice of Butter-making	8vo,	*1 50
MAYNARD—Landscape Gardening as Applied to Home Decoration	12mo,	1 50
MOON and BROWN—Elements of Forestry	(<i>In Preparation.</i>)	
RECORD—Identification of the Economic Woods of the United States	8vo,	*1 25
RECKNAGEL—Theory and Practice of Working Plans (Forest Organization) 8vo,		*2 00

Article 11

PROJECT GUTENBERG “SMALL PRINT”

End of the Project Gutenberg EBook of Vector Analysis and Quaternions,
by Alexander Macfarlane

*** END OF THIS PROJECT GUTENBERG EBOOK VECTOR ANALYSIS AND QUATERNIONS ***

***** This file should be named 13609-pdf.pdf or 13609-pdf.zip *****
This and all associated files of various formats will be found in:
<http://www.gutenberg.net/1/3/6/0/13609/>

Produced by David Starner, Joshua Hutchinson, John Hagerson, and the
Project Gutenberg On-line Distributed Proofreaders.

Updated editions will replace the previous one--the old editions
will be renamed.

Creating the works from public domain print editions means that no
one owns a United States copyright in these works, so the Foundation
(and you!) can copy and distribute it in the United States without
permission and without paying copyright royalties. Special rules,
set forth in the General Terms of Use part of this license, apply to
copying and distributing Project Gutenberg-tm electronic works to
protect the PROJECT GUTENBERG-tm concept and trademark. Project
Gutenberg is a registered trademark, and may not be used if you
charge for the eBooks, unless you receive specific permission. If
you do not charge anything for copies of this eBook, complying with
the rules is very easy. You may use this eBook for nearly any
purpose such as creation of derivative works, reports, performances
and research. They may be modified and printed and given away--you

ARTICLE 11. PROJECT GUTENBERG "SMALL PRINT"

may do practically ANYTHING with public domain eBooks.
Redistribution is subject to the trademark license, especially
commercial redistribution.

*** START: FULL LICENSE ***

THE FULL PROJECT GUTENBERG LICENSE PLEASE READ THIS BEFORE YOU
DISTRIBUTE OR USE THIS WORK

To protect the Project Gutenberg-tm mission of promoting the free
distribution of electronic works, by using or distributing this work
(or any other work associated in any way with the phrase "Project
Gutenberg"), you agree to comply with all the terms of the Full
Project Gutenberg-tm License (available with this file or online at
<http://gutenberg.net/license>).

Section 1. General Terms of Use and Redistributing Project
Gutenberg-tm electronic works

1.A. By reading or using any part of this Project Gutenberg-tm
electronic work, you indicate that you have read, understand, agree
to and accept all the terms of this license and intellectual
property (trademark/copyright) agreement. If you do not agree to
abide by all the terms of this agreement, you must cease using and
return or destroy all copies of Project Gutenberg-tm electronic
works in your possession. If you paid a fee for obtaining a copy of
or access to a Project Gutenberg-tm electronic work and you do not
agree to be bound by the terms of this agreement, you may obtain a
refund from the person or entity to whom you paid the fee as set
forth in paragraph 1.E.8.

1.B. "Project Gutenberg" is a registered trademark. It may only be
used on or associated in any way with an electronic work by people
who agree to be bound by the terms of this agreement. There are a
few things that you can do with most Project Gutenberg-tm electronic
works even without complying with the full terms of this agreement.
See paragraph 1.C below. There are a lot of things you can do with
Project Gutenberg-tm electronic works if you follow the terms of
this agreement and help preserve free future access to Project
Gutenberg-tm electronic works. See paragraph 1.E below.

1.C. The Project Gutenberg Literary Archive Foundation ("the
Foundation" or PGLAF), owns a compilation copyright in the
collection of Project Gutenberg-tm electronic works. Nearly all the
individual works in the collection are in the public domain in the
United States. If an individual work is in the public domain in the
United States and you are located in the United States, we do not

ARTICLE 11. PROJECT GUTENBERG "SMALL PRINT"

claim a right to prevent you from copying, distributing, performing, displaying or creating derivative works based on the work as long as all references to Project Gutenberg are removed. Of course, we hope that you will support the Project Gutenberg-tm mission of promoting free access to electronic works by freely sharing Project Gutenberg-tm works in compliance with the terms of this agreement for keeping the Project Gutenberg-tm name associated with the work. You can easily comply with the terms of this agreement by keeping this work in the same format with its attached full Project Gutenberg-tm License when you share it without charge with others.

1.D. The copyright laws of the place where you are located also govern what you can do with this work. Copyright laws in most countries are in a constant state of change. If you are outside the United States, check the laws of your country in addition to the terms of this agreement before downloading, copying, displaying, performing, distributing or creating derivative works based on this work or any other Project Gutenberg-tm work. The Foundation makes no representations concerning the copyright status of any work in any country outside the United States.

1.E. Unless you have removed all references to Project Gutenberg:

1.E.1. The following sentence, with active links to, or other immediate access to, the full Project Gutenberg-tm License must appear prominently whenever any copy of a Project Gutenberg-tm work (any work on which the phrase "Project Gutenberg" appears, or with which the phrase "Project Gutenberg" is associated) is accessed, displayed, performed, viewed, copied or distributed:

This eBook is for the use of anyone anywhere at no cost and with almost no restrictions whatsoever. You may copy it, give it away or re-use it under the terms of the Project Gutenberg License included with this eBook or online at www.gutenberg.net

1.E.2. If an individual Project Gutenberg-tm electronic work is derived from the public domain (does not contain a notice indicating that it is posted with permission of the copyright holder), the work can be copied and distributed to anyone in the United States without paying any fees or charges. If you are redistributing or providing access to a work with the phrase "Project Gutenberg" associated with or appearing on the work, you must comply either with the requirements of paragraphs 1.E.1 through 1.E.7 or obtain permission for the use of the work and the Project Gutenberg-tm trademark as set forth in paragraphs 1.E.8 or 1.E.9.

1.E.3. If an individual Project Gutenberg-tm electronic work is posted with the permission of the copyright holder, your use and distribution must comply with both paragraphs 1.E.1 through 1.E.7 and any additional terms imposed by the copyright holder.

ARTICLE 11. PROJECT GUTENBERG "SMALL PRINT"

Additional terms will be linked to the Project Gutenberg-tm License for all works posted with the permission of the copyright holder found at the beginning of this work.

1.E.4. Do not unlink or detach or remove the full Project Gutenberg-tm License terms from this work, or any files containing a part of this work or any other work associated with Project Gutenberg-tm.

1.E.5. Do not copy, display, perform, distribute or redistribute this electronic work, or any part of this electronic work, without prominently displaying the sentence set forth in paragraph 1.E.1 with active links or immediate access to the full terms of the Project Gutenberg-tm License.

1.E.6. You may convert to and distribute this work in any binary, compressed, marked up, nonproprietary or proprietary form, including any word processing or hypertext form. However, if you provide access to or distribute copies of a Project Gutenberg-tm work in a format other than "Plain Vanilla ASCII" or other format used in the official version posted on the official Project Gutenberg-tm web site (www.gutenberg.net), you must, at no additional cost, fee or expense to the user, provide a copy, a means of exporting a copy, or a means of obtaining a copy upon request, of the work in its original "Plain Vanilla ASCII" or other form. Any alternate format must include the full Project Gutenberg-tm License as specified in paragraph 1.E.1.

1.E.7. Do not charge a fee for access to, viewing, displaying, performing, copying or distributing any Project Gutenberg-tm works unless you comply with paragraph 1.E.8 or 1.E.9.

1.E.8. You may charge a reasonable fee for copies of or providing access to or distributing Project Gutenberg-tm electronic works provided that

- You pay a royalty fee of 20% of the gross profits you derive from the use of Project Gutenberg-tm works calculated using the method you already use to calculate your applicable taxes. The fee is owed to the owner of the Project Gutenberg-tm trademark, but he has agreed to donate royalties under this paragraph to the Project Gutenberg Literary Archive Foundation. Royalty payments must be paid within 60 days following each date on which you prepare (or are legally required to prepare) your periodic tax returns. Royalty payments should be clearly marked as such and sent to the Project Gutenberg Literary Archive Foundation at the address specified in Section 4, "Information about donations to the Project Gutenberg Literary Archive Foundation."
- You provide a full refund of any money paid by a user who notifies

ARTICLE 11. PROJECT GUTENBERG "SMALL PRINT"

you in writing (or by e-mail) within 30 days of receipt that s/he does not agree to the terms of the full Project Gutenberg-tm License. You must require such a user to return or destroy all copies of the works possessed in a physical medium and discontinue all use of and all access to other copies of Project Gutenberg-tm works.

- You provide, in accordance with paragraph 1.F.3, a full refund of any money paid for a work or a replacement copy, if a defect in the electronic work is discovered and reported to you within 90 days of receipt of the work.
- You comply with all other terms of this agreement for free distribution of Project Gutenberg-tm works.

1.E.9. If you wish to charge a fee or distribute a Project Gutenberg-tm electronic work or group of works on different terms than are set forth in this agreement, you must obtain permission in writing from both the Project Gutenberg Literary Archive Foundation and Michael Hart, the owner of the Project Gutenberg-tm trademark. Contact the Foundation as set forth in Section 3 below.

1.F.

1.F.1. Project Gutenberg volunteers and employees expend considerable effort to identify, do copyright research on, transcribe and proofread public domain works in creating the Project Gutenberg-tm collection. Despite these efforts, Project Gutenberg-tm electronic works, and the medium on which they may be stored, may contain "Defects," such as, but not limited to, incomplete, inaccurate or corrupt data, transcription errors, a copyright or other intellectual property infringement, a defective or damaged disk or other medium, a computer virus, or computer codes that damage or cannot be read by your equipment.

1.F.2. LIMITED WARRANTY, DISCLAIMER OF DAMAGES - Except for the "Right of Replacement or Refund" described in paragraph 1.F.3, the Project Gutenberg Literary Archive Foundation, the owner of the Project Gutenberg-tm trademark, and any other party distributing a Project Gutenberg-tm electronic work under this agreement, disclaim all liability to you for damages, costs and expenses, including legal fees. YOU AGREE THAT YOU HAVE NO REMEDIES FOR NEGLIGENCE, STRICT LIABILITY, BREACH OF WARRANTY OR BREACH OF CONTRACT EXCEPT THOSE PROVIDED IN PARAGRAPH F3. YOU AGREE THAT THE FOUNDATION, THE TRADEMARK OWNER, AND ANY DISTRIBUTOR UNDER THIS AGREEMENT WILL NOT BE LIABLE TO YOU FOR ACTUAL, DIRECT, INDIRECT, CONSEQUENTIAL, PUNITIVE OR INCIDENTAL DAMAGES EVEN IF YOU GIVE NOTICE OF THE POSSIBILITY OF SUCH DAMAGE.

1.F.3. LIMITED RIGHT OF REPLACEMENT OR REFUND - If you discover a

ARTICLE 11. PROJECT GUTENBERG "SMALL PRINT"

defect in this electronic work within 90 days of receiving it, you can receive a refund of the money (if any) you paid for it by sending a written explanation to the person you received the work from. If you received the work on a physical medium, you must return the medium with your written explanation. The person or entity that provided you with the defective work may elect to provide a replacement copy in lieu of a refund. If you received the work electronically, the person or entity providing it to you may choose to give you a second opportunity to receive the work electronically in lieu of a refund. If the second copy is also defective, you may demand a refund in writing without further opportunities to fix the problem.

1.F.4. Except for the limited right of replacement or refund set forth in paragraph 1.F.3, this work is provided to you 'AS-IS', WITH NO OTHER WARRANTIES OF ANY KIND, EXPRESS OR IMPLIED, INCLUDING BUT NOT LIMITED TO WARRANTIES OF MERCHANTABILITY OR FITNESS FOR ANY PURPOSE.

1.F.5. Some states do not allow disclaimers of certain implied warranties or the exclusion or limitation of certain types of damages. If any disclaimer or limitation set forth in this agreement violates the law of the state applicable to this agreement, the agreement shall be interpreted to make the maximum disclaimer or limitation permitted by the applicable state law. The invalidity or unenforceability of any provision of this agreement shall not void the remaining provisions.

1.F.6. INDEMNITY - You agree to indemnify and hold the Foundation, the trademark owner, any agent or employee of the Foundation, anyone providing copies of Project Gutenberg-tm electronic works in accordance with this agreement, and any volunteers associated with the production, promotion and distribution of Project Gutenberg-tm electronic works, harmless from all liability, costs and expenses, including legal fees, that arise directly or indirectly from any of the following which you do or cause to occur: (a) distribution of this or any Project Gutenberg-tm work, (b) alteration, modification, or additions or deletions to any Project Gutenberg-tm work, and (c) any Defect you cause.

Section 2. Information about the Mission of Project Gutenberg-tm

Project Gutenberg-tm is synonymous with the free distribution of electronic works in formats readable by the widest variety of computers including obsolete, old, middle-aged and new computers. It exists because of the efforts of hundreds of volunteers and donations from people in all walks of life.

Volunteers and financial support to provide volunteers with the

ARTICLE 11. PROJECT GUTENBERG "SMALL PRINT"

assistance they need, is critical to reaching Project Gutenberg-tm's goals and ensuring that the Project Gutenberg-tm collection will remain freely available for generations to come. In 2001, the Project Gutenberg Literary Archive Foundation was created to provide a secure and permanent future for Project Gutenberg-tm and future generations. To learn more about the Project Gutenberg Literary Archive Foundation and how your efforts and donations can help, see Sections 3 and 4 and the Foundation web page at <http://www.pglaf.org>.

Section 3. Information about the Project Gutenberg Literary Archive Foundation

The Project Gutenberg Literary Archive Foundation is a non profit 501(c)(3) educational corporation organized under the laws of the state of Mississippi and granted tax exempt status by the Internal Revenue Service. The Foundation's EIN or federal tax identification number is 64-6221541. Its 501(c)(3) letter is posted at <http://pglaf.org/fundraising>. Contributions to the Project Gutenberg Literary Archive Foundation are tax deductible to the full extent permitted by U.S. federal laws and your state's laws.

The Foundation's principal office is located at 4557 Melan Dr. S. Fairbanks, AK, 99712., but its volunteers and employees are scattered throughout numerous locations. Its business office is located at 809 North 1500 West, Salt Lake City, UT 84116, (801) 596-1887, email business@pglaf.org. Email contact links and up to date contact information can be found at the Foundation's web site and official page at <http://pglaf.org>

For additional contact information:

Dr. Gregory B. Newby
Chief Executive and Director
gnewby@pglaf.org

Section 4. Information about Donations to the Project Gutenberg Literary Archive Foundation

Project Gutenberg-tm depends upon and cannot survive without wide spread public support and donations to carry out its mission of increasing the number of public domain and licensed works that can be freely distributed in machine readable form accessible by the widest array of equipment including outdated equipment. Many small donations (\$1 to \$5,000) are particularly important to maintaining tax exempt status with the IRS.

The Foundation is committed to complying with the laws regulating charities and charitable donations in all 50 states of the United States. Compliance requirements are not uniform and it takes a

ARTICLE 11. PROJECT GUTENBERG "SMALL PRINT"

considerable effort, much paperwork and many fees to meet and keep up with these requirements. We do not solicit donations in locations where we have not received written confirmation of compliance. To SEND DONATIONS or determine the status of compliance for any particular state visit <http://pglaf.org>

While we cannot and do not solicit contributions from states where we have not met the solicitation requirements, we know of no prohibition against accepting unsolicited donations from donors in such states who approach us with offers to donate.

International donations are gratefully accepted, but we cannot make any statements concerning tax treatment of donations received from outside the United States. U.S. laws alone swamp our small staff.

Please check the Project Gutenberg Web pages for current donation methods and addresses. Donations are accepted in a number of other ways including including checks, online payments and credit card donations. To donate, please visit: <http://pglaf.org/donate>

Section 5. General Information About Project Gutenberg-tm electronic works.

Professor Michael S. Hart is the originator of the Project Gutenberg-tm concept of a library of electronic works that could be freely shared with anyone. For thirty years, he produced and distributed Project Gutenberg-tm eBooks with only a loose network of volunteer support.

Project Gutenberg-tm eBooks are often created from several printed editions, all of which are confirmed as Public Domain in the U.S. unless a copyright notice is included. Thus, we do not necessarily keep eBooks in compliance with any particular paper edition.

Most people start at our Web site which has the main PG search facility:

<http://www.gutenberg.net>

This Web site includes information about Project Gutenberg-tm, including how to make donations to the Project Gutenberg Literary Archive Foundation, how to help produce our new eBooks, and how to subscribe to our email newsletter to hear about new eBooks.